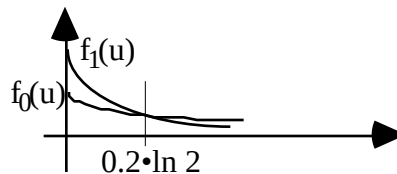


- 1.(a) As is obvious from the figure, the chord is longer than the side of the inscribed equilateral triangle if $2\pi/3 < X < 4\pi/3$. Hence, the desired probability is $(4\pi/3 - 2\pi/3)/2\pi = 1/3$ as in the second model in Ross. What is the geometrical relation between the two models?
- (b) Since the circle has radius 1, an arc of length θ subtends an angle θ at C. Also, the length of the chord joining the endpoints of the arc is $2 \sin(\theta/2)$. Hence, $L = 2 \sin(X/2)$. Note that as X increases from 0 to 2π , the chord length increases from 0 to 2 (at $X = \pi$), and then decreases to 0 (at $X = 2\pi$). For any x , $0 \leq x \leq 2$, $F_L(x) = P\{L \leq x\} = P\{2 \sin(X/2) \leq x\} = 2P\{0 \leq X \leq 2 \arcsin(x/2)\}$ (Why twice?) $= 2(2 \arcsin(x/2)/2\pi) = (2/\pi)\arcsin(x/2)$.

$$\text{Hence, } f_L(x) = \frac{d}{dx} F_L(x) = \begin{cases} \frac{1}{\pi \sqrt{1-(x/2)^2}} & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- 2.(a) Obviously $P\{Y = \alpha\} = P\{Y = -\alpha\} = 1/2$.
- (b) $(1.29-1) = 0.29$. $(1.29-1)^2 = 0.0841$. $(\pi/4-1) = -0.214\dots$, $(\pi/4-1)^2 = 0.046\dots$.
 $(-\pi/4-(-1)) = -0.214\dots$, $(-\pi/4-(-1))^2 = 0.046\dots$. Note that the error for $+X$ is the same as that for $-X$.
- (c) $E[Z] = \int_{-\infty}^{\infty} (u-\alpha)^2 f(u) du + \int_{-\infty}^{\infty} (u+\alpha)^2 f(u) du = \int_{-\infty}^{\infty} (u^2 + \alpha^2) f(u) du - 4\alpha \int_{-\infty}^{\infty} u f(u) du = 1 + \alpha^2 - 4\alpha \sqrt{2\pi}$ on expanding out the quadratics, changing variables, and using the fact that $E[X^2] = \sigma^2 + \mu^2 = 1$. Note that $uf(u)$ is a perfect integral. It is easy to show that $E[Z]$ has minimum value $1-2/\pi$ at $\alpha = \sqrt{2/\pi}$.
- (d) From tables of $\Phi(\cdot)$, we get $P\{W = -3\} = P\{W = +3\} = \Phi(-2.5) = 0.0062$,
 $P\{W = 0\} = \Phi(0.5) - \Phi(-0.5) = 0.3830$, $P\{W = -1\} = P\{W = +1\} = \Phi(1.5) - \Phi(0.5) = 0.2417$, and
 $P\{W = -2\} = P\{W = +2\} = \Phi(2.5) - \Phi(1.5) = 0.0606$.
- (e) $P\{Z_2 = 1\} = P\{W < 0\} = 0.3085$.
 $P\{Z_1 = 1\} = P\{W = 2\} + P\{W = 3\} + P\{W = -1\} + P\{W = -2\} = 0.3691$
 $P\{Z_0 = 0\} = P\{W = 2\} + P\{W = 0\} + P\{W = -2\} = 0.5042 \neq P\{Z_0 = 1\}$
3. $\frac{d}{db}(bt)\exp(-bt^2/2) = t \cdot \exp(-bt^2/2) - bt \cdot \exp(-bt^2/2) \cdot t/2$ is zero for $b = \sqrt{2}/t$. Thus, if we observe that $X = t$, the maximum-likelihood estimate of b is $\sqrt{2}/t$. Reality check: If the observed value t is large, we estimate the value of b to be quite small. This makes sense. If the system lasted for a long time, its hazard rate can be expected to be small, and the hazard rate is proportional to b .

4.(a)



- (b) The maximum-likelihood decision chooses the hypothesis which has the larger pdf value at the observation. Since the two pdfs cross at $\theta = 0.2 \cdot \ln 2$, we see that the decision is to choose H_0 if $X > \theta$ and H_1 if $X < \theta$.
- (c) $P_{FA} = P\{\text{false alarm}\} = P\{H_1 \text{ is chosen when in fact } H_0 \text{ is the true hypothesis}\} = P\{X < \theta \text{ when } H_0 \text{ is true}\}$
 $= \int_0^{\theta} f_0(u) du = \int_0^{\theta} 5 \cdot \exp(-5u) du = 1 - \exp(-5\theta) = 1/2$.
- $P_{MD} = P\{\text{missed detection}\} = P\{X > \theta \text{ when } H_1 \text{ is true}\} = \int_{\theta}^{\infty} f_1(u) du = \int_{\theta}^{\infty} 10 \cdot \exp(-10u) du = \exp(-10\theta) = 1/4$.

- (d) $\Lambda(u) = \frac{10 \cdot \exp(-10u)}{5 \cdot \exp(-5u)} = 2 \cdot \exp(-5u)$ decreases from 2 at $u = 0$ to 1 at $u = 0.2 \cdot \ln 2$ to 0 at $u = \infty$. We compare $\Lambda(u)$ to π_0/π_1 to get that $\Lambda(u) > \pi_0/\pi_1$ if $u < -0.2 \cdot \ln(\pi_0/2\pi_1) = \xi$, that is, the Bayesian rule chooses H_0 if $\mathbf{X} > \xi$ and H_1 if $\mathbf{X} < \xi$. Note that $\xi < 0$ if $\pi_0 > 2\pi_1$.
- (e) Now $\xi = -0.2 \cdot \ln(1/4) = 0.2 \cdot \ln(4)$. Proceeding as in part (c), we readily get that $P_{FA} = 1 - \exp(-5\xi) = 3/4$ while $P_{MD} = \exp(-10\xi) = 1/16$. The average error probability is thus $(3/4) \cdot (1/3) + (1/16) \cdot (2/3) = 7/24$. Notice that since $\pi_0 < \pi_1$, the Bayesian decision rule accepts a larger P_{FA} in return for a smaller P_{MD} because P_{FA} has a smaller weight than P_{MD} in the expression for the average error probability.
- (f) The decision rule that always chooses H_0 makes an error precisely in those instances when H_1 is the true hypothesis. Hence its average error probability is just π_1 , the probability that H_1 is the true hypothesis.
- (h) The Bayesian decision rule compares the likelihood ratio $\Lambda(u) = 2 \cdot \exp(-5u)$ to π_0/π_1 . But, since $\Lambda(u)$ has maximum value 2, it is always smaller than π_0/π_1 if $\pi_0 > 2\pi_1$, that is, if $\pi_0 > 2/3$. It thus achieves average error probability $\pi_1 < 1/3$. In contrast, for these prior probabilities, the ML rule achieves an average error probability $(1/2) \cdot \pi_0 + (1/4) \cdot \pi_1 > (1/2) \cdot (2\pi_1) + (1/4) \cdot \pi_1 = 1.25\pi_1$.

- 5.(a) No, the unconditional pdf of $\mathbf{X}$ is given by $[(a\sqrt{2\pi})^{-1} \exp(-u^2/2a^2) + (b\sqrt{2\pi})^{-1} \exp(-u^2/2b^2)]/2$, which is not a Gaussian pdf.

(b)
$$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{(a\sqrt{2\pi}) \exp(-u^2/2b^2)}{(b\sqrt{2\pi}) \exp(-u^2/2a^2)} = \frac{a}{b} \cdot \exp \frac{-u^2}{2} \frac{1}{b^2} - \frac{1}{a^2} u^2$$

- (c) Suppose that the observation $\mathbf{X}$ has value u . The maximum-likelihood decision rule says that H_1 is chosen as the true hypothesis if $\Lambda(u) > 1$ and H_0 is chosen if $\Lambda(u) < 1$. Thus, H_1 is chosen if

$$\ln(a/b) - (u^2/2)(b^{-2} - a^{-2}) > 0.$$

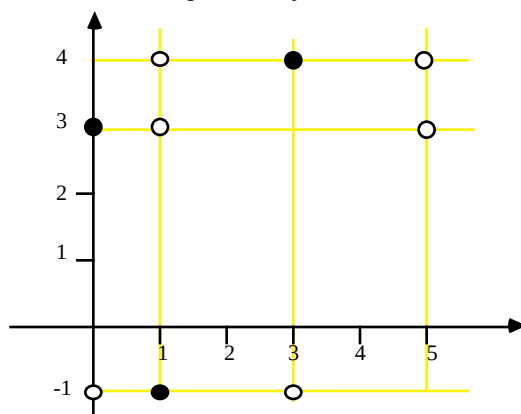
This is equivalent to the statement that the rule chooses H_1 whenever the observation $\mathbf{X}$ is such that

$$|\mathbf{X}| > ab \sqrt{\frac{\ln b^2 - \ln a^2}{b^2 - a^2}} = c.$$

Note that $f_0(0) = 1/(a\sqrt{2\pi}) > 1/(b\sqrt{2\pi}) = f_1(0)$ and the two pdf curves cross each other at $\pm c$.

$P(\text{false alarm}) = P\{|\mathbf{X}| > c \mid H_0 \text{ is true}\} = 2Q(c/a)$. $P(\text{missed detection}) = P\{|\mathbf{X}| < c \mid H_1 \text{ is true}\} = 1 - 2Q(c/b)$.

- 6.(a) In the figure below, I use $\circ$ to denote a probability mass of $1/12$ and $\bullet$ to denote a probability mass of $1/6$.



- (a) $p_{\mathbf{X}}(0) = 1/6 + 1/12 = 1/4$, $p_{\mathbf{X}}(1) = 1/3$, $p_{\mathbf{X}}(3) = 1/4$, $p_{\mathbf{X}}(5) = 1/6$.
 $p_{\mathbf{Y}}(-1) = 2 \times 1/12 + 1/6 = 1/3$, $p_{\mathbf{Y}}(3) = 1/3$, $p_{\mathbf{Y}}(4) = 1/3$.
- (b) Since, for example, $p_{\mathbf{X}}(3)p_{\mathbf{Y}}(3) = (1/3) \times (1/4) = 1/12 \neq p_{\mathbf{X}, \mathbf{Y}}(3, 3) = 0$, the random variables are not independent. We can also see this via the eyeball test: there are empty spots in the grid.
- (c) $P\{\mathbf{X} \leq \mathbf{Y}\} = p_{\mathbf{X}, \mathbf{Y}}(0, 3) + p_{\mathbf{X}, \mathbf{Y}}(1, 3) + p_{\mathbf{X}, \mathbf{Y}}(1, 4) + p_{\mathbf{X}, \mathbf{Y}}(3, 4) = 1/2$.
 $P\{\mathbf{X} + \mathbf{Y} \leq 8\} = 1 - p_{\mathbf{X}, \mathbf{Y}}(5, 4) = 11/12$.
- (d) Given $\mathbf{Y} = 3$, $\mathbf{X}$ takes on values 0, 1 and 5 with conditional probabilities $(1/6)/(1/3)$, $(1/12)/(1/3)$, and $(1/12)/(1/3)$, that is, $1/2$, $1/4$ and $1/4$ respectively. Hence, $E[\mathbf{X}|\mathbf{Y}=3] = 0 \cdot (1/2) + 1 \cdot (1/4) + 5 \cdot (1/4) = 3/2$, and $E[\mathbf{X}^2|\mathbf{Y}=3] = 0^2 \cdot (1/2) + 1^2 \cdot (1/4) + 5^2 \cdot (1/4) = 26/4$, giving $\text{var}(\mathbf{X}|\mathbf{Y}=3) = 26/4 - 9/4 = 17/4$.