

- 1.(a) $d/du(\exp(-u^2/2)) = -u \exp(-u^2/2)$. Hence,

$$E[|X|] = \int_{-\infty}^{\infty} |u| \cdot \phi(u) du = 2 \int_0^{\infty} u \cdot \phi(u) du = 2 \int_0^{\infty} u \cdot \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) du = \sqrt{\frac{2}{\pi}} \cdot \exp(-u^2/2) \Big|_0^{\infty} = \sqrt{\frac{2}{\pi}} < 1.$$

More generally, $E[|X-\mu|] = \sigma \sqrt{\frac{2}{\pi}}$ if the pdf of X is $N(\mu, \sigma^2)$. In statistical applications, $E[|X-\mu|]$ is sometimes called the absolute error.

- (b) Since $\Phi(x) = 0.9999\dots$ etc for large x , the noninformative 9's waste paper, whereas $Q(x)$ can be specified more easily as, for example, $Q(5) = 2.8665 \times 10^{-7}$.

- (c) $Q(x) = \int_x^{\infty} \phi(u) du = \int_x^{\infty} u^{-1} u \phi(u) du = x^{-1} \phi(x) - \int_x^{\infty} u^{-2} \phi(u) du = x^{-1} \phi(x) - \int_x^{\infty} u^{-3} u \phi(u) du$
 $= x^{-1} \phi(x) - x^{-3} \phi(x) + \int_x^{\infty} 3u^{-4} \phi(u) du$. Since all the integrands are positive, so are all the integrals, and it follows that

$$x^{-1} \phi(x) - x^{-3} \phi(x) < Q(x) < x^{-1} \phi(x)$$

which is Eq. (4.4) of Ross. Clearly, the bounds tend to $-\infty$ and $+\infty$ as $x \rightarrow 0$ and thus are not useful for small x . On the other hand, the **ratio** of the upper and lower bounds on $Q(x)$ is $1 + (x^2 - 1)^{-1}$ which approaches 1 as $x \rightarrow \infty$. Note that if either bound is used as an approximation for the value of $Q(x)$, the maximum relative error is also **guaranteed** to decrease as x increases. For example, at $x = 5$, the maximum relative error is $1/24 \approx 4.2\%$.

- (d) Since $t + x > t - x > 0$, $(t + x)(t - x) = t^2 - x^2 > (t - x)^2$. Hence, $\exp(-(t^2 - x^2)) < \exp(-(t - x)^2)$ and therefore $\exp\left(\frac{x^2}{2}\right) Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2 - x^2}{2}\right) dt < \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(t - x)^2}{2}\right) dt = 1/2$. What? Why 1/2?

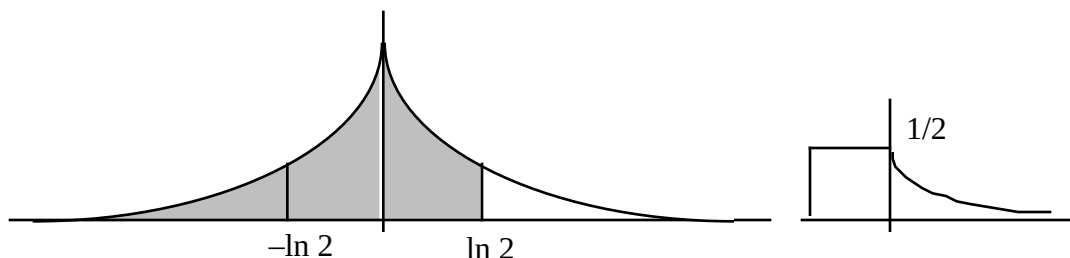
- (e) The bound $(1/2)\exp(-x^2/2)$ is tighter for $x < \sqrt{2/\pi} \approx 0.7979$.

- 2.(b)(c) Eq. 26.2.17 is the most commonly used approximation to $Q(x)$. It gives $Q(5) \approx 2.8710 \times 10^{-7}$. The error in the approximation is known to be smaller than 7.5×10^{-8} so that the actual value of $Q(5)$ *could be* as small as $2.8710 \times 10^{-7} - 7.5 \times 10^{-8} = 2.1210 \times 10^{-7}$ or it *could be* as large as $2.8710 \times 10^{-7} + 7.5 \times 10^{-8} = 3.6210 \times 10^{-7}$ and thus the maximum relative error *could be* as large as $7.5 \times 10^{-8} / 2.1210 \times 10^{-7} \approx 35.4\%$.

- (d) $-\log_{10} Q(5) = 6.54265$, so $Q(5) = 2.86675 \times 10^{-7}$. Thus, even though the error specification implies that there *could be* a 35% error, Eq. 26.2.17 actually overestimates $Q(5)$ by only 0.16%! In contrast, the bounds $(1/5 - 1/5^3) \phi(5) < Q(5) < \phi(5)/5$ give $2.8545 \times 10^{-7} < Q(5) < 2.9734 \times 10^{-7}$. The upper bound overestimates $Q(5)$ by 3.57% and the lower bound underestimates $Q(5)$ by 0.42%. Thus, at $x = 5$, both the upper bound and the lower bound are poorer approximations to $Q(5)$ than the value given by Eq. (26.2.17).

- (e) The maximum possible error is larger than the value being computed.

3. $Y(f) = H(f)X(f) = 2 \cdot \exp(-\pi f^2)$ for $|f| \leq 1$, and hence $y(0) = \int_{-\infty}^{\infty} Y(f) \cdot \exp(j2\pi f \cdot 0) df = \int_{-1}^1 2 \cdot \exp(-\pi f^2) df$
 $= 2 \cdot [\Phi(\sqrt{2\pi}) - \Phi(-\sqrt{2\pi})] = 2 \cdot [2 \cdot \Phi(\sqrt{2\pi}) - 1] = 2 \cdot [2 \cdot \Phi(2.38\dots) - 1] = 1.9652\dots$



- 4.(a) From the figure shown above, we see that the pdf is symmetric about $u = 0$. Hence, we get that

$$P\{X \leq \ln 2\} = \frac{1}{2} + \int_0^{\ln 2} \frac{1}{2} \exp(-u) du = \frac{1}{2} + \left[-\frac{1}{2} \exp(-u) \right]_0^{\ln 2} = \frac{3}{4}. \text{ Notice that } P\{0 \leq X \leq \ln 2\} = \frac{1}{4}$$

$$(b) \quad P\{|X| \leq \ln 2 \mid X \leq \ln 2\} = \frac{P\{|X| \leq \ln 2\} \cap \{X \leq \ln 2\}}{P\{X \leq \ln 2\}} = \frac{P\{|X| \leq \ln 2\}}{P\{X \leq \ln 2\}} = \frac{2P\{0 \leq X \leq \ln 2\}}{3/4} = \frac{1/2}{3/4} = \frac{2}{3}.$$

$$(c) \quad \cos(\pi X/2) = 0 \text{ if } X \text{ is an odd integer, and} \\ P\{\cos(\pi X/2) < 0\} = \dots + P\{-7 < X < -5\} + P\{-3 < X < -1\} + P\{1 < X < 3\} + P\{5 < X < 7\} + \dots \\ = 2[P\{1 < X < 3\} + P\{5 < X < 7\} + \dots] \text{ (by symmetry)} = [\exp(-1) - \exp(-3)] + [\exp(-5) - \exp(-7)] + \dots \\ = \exp(-1) \cdot \frac{1}{1 + \exp(-2)} = \frac{1}{2 \cosh(1)} = 0.324\dots \text{ on summing the geometric series.}$$

$$(d) \quad \text{The minimum value of } I \text{ is } -1. \quad F_I(b) = P\{I \leq b\} = 0 \text{ if } b < -1. \\ \text{For } b \geq -1, F_I(b) = P\{I \leq b\} = P\{e^V - 1 \leq b\} = P\{V \leq \ln(b+1)\} = F_V(\ln(b+1)). \text{ But,}$$

$$F_V(a) = \begin{cases} e^{a/2}, & a \leq 0, \\ 1 - e^{-a/2}, & a > 0. \end{cases} \quad \text{Thus, } F_I(b) = \begin{cases} \frac{b+1}{2}, & -1 \leq b \leq 0, \\ 1 - \frac{1}{2(b+1)}, & b > 0, \end{cases}$$

$$\text{and } f_I(b) = \begin{cases} \frac{1}{2}, & -1 \leq b \leq 0, \\ \frac{1}{2(b+1)^2}, & b > 0. \end{cases} \quad \text{Note that the pdf has constant value } 1/2 \text{ in the range } -1 \leq b \leq 0.$$

$$5.(a) \quad \text{Let } 0 \leq v \leq 1. \text{ Then, } F_Y(v) = P\{Y \leq v\} = P\{(1-X)^2 \leq v\} = P\{-\sqrt{v} \leq 1-X \leq \sqrt{v}\} = P\{X \geq 1-\sqrt{v}\} \\ = 1 - F_X(1-\sqrt{v}) = (1 - (1-\sqrt{v}))^2 = v \text{ where we used the result that } F_X(u) = 1 - (1-u)^2.$$

$$\text{Hence, } F_Y(v) = \begin{cases} 0, & v < 0, \\ v, & 0 \leq v \leq 1, \\ 1, & v > 1. \end{cases}$$

$$(b) \quad \text{A sketch of the function } F_Y(v) \text{ reveals that it is a nondecreasing continuous function. It is not differentiable at } \alpha = 0 \text{ or at } \beta = 1.$$

$$6.(a) \quad E[R] = \int_0^1 3\rho^3 d\rho = \frac{3}{4}; \quad E[V] = E[4\pi R^3/3] = \int_0^1 4\pi \rho^5 d\rho = \frac{2\pi}{3}; \quad E[A] = E[4\pi R^2] = \int_0^1 12\pi \rho^4 d\rho = \frac{12\pi}{5}.$$

The average volume $E[V] = E[4\pi R^3/3]$ corresponds to a sphere of radius $(1/2)^{1/3}$ and the average surface area $E[A] = E[4\pi R^2]$ to a sphere of radius $(3/5)^{1/2}$. Note that $E[V] = E[4\pi R^3/3] \neq 4\pi(E[R])^3/3$, etc. This illustrates the general result that $E[g(X)]$ hardly ever equals $g(E[X])$. You will save yourself a lot of grief if you keep this in mind: that $E[g(X)] = g(E[X])$ is a common misconception among the instochaste.

Exercise: Find a function $g(\bullet)$ for which $E[g(X)]$ does equal $g(E[X])$.

$$(b) \quad \text{The volume } V \text{ has values in the range } (0, 4\pi/3). \text{ For any } u, 0 < u < 4\pi/3, F_V(u) = P\{V \leq u\} \\ = P\{4\pi R^3/3 \leq u\} = P\{R \leq \sqrt[3]{3u/4\pi}\} = F_R(\sqrt[3]{3u/4\pi}) = 3u/4\pi \text{ since } F_R(\rho) = \rho^3 \text{ for } 0 < \rho < 1. \text{ Hence, } f_V(u) \text{ is uniform on } (0, 4\pi/3).$$

$$(c) \quad \text{Obviously, } E[V] = \text{midpoint of uniform pdf} = 2\pi/3 \text{ as in part (a)}$$

$$(d) \quad \text{The electrical charge is uniformly distributed on the surface of the sphere. The surface charge density is } S = Q/4\pi R^2 > Q/4\pi.$$

$$\text{For } x > Q/4\pi, F_S(x) = P\{S \leq x\} = P\{Q/4\pi R^2 \leq x\} = P\{1 > R \geq \sqrt{Q/4\pi x}\} = 1 - (Q/4\pi x)^{1.5}.$$

$$\text{Hence, } f_S(x) = (3/2x)(Q/4\pi x)^{1.5} \text{ for } x > Q/4\pi, \text{ and } 0 \text{ otherwise.}$$

$$7. \quad \text{Assuming that life begins at forty (ask your parents about this!), the probability that the smoker survives to age 50 without getting lung cancer is } P\{X > 10\} \text{ where } X \text{ is the number of years that he lives cancer-free}$$

$$\text{beyond 40. The hazard rate of } X \text{ is } h(t) = 0.027 + 0.00025t^2. \text{ Thus, } P\{X > 10\} = \exp\left(-\int_0^{10} h(t) dt\right) = \exp(-1.06/3) \approx \exp(-0.3533\dots) = 0.70\dots$$

$$\text{Similarly, } P\{X > 20\} = \exp\left(-\int_0^{20} h(t) dt\right) = \exp(-3.62/3) \approx \exp(-1.20666\dots) = 0.30\dots$$