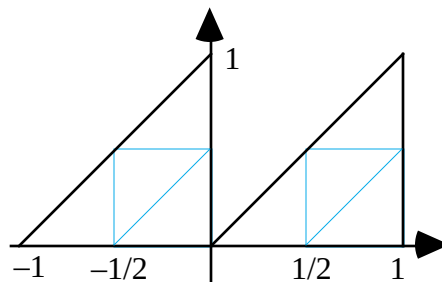


- 1.(a) This is a valid pdf. (b) This is a valid pdf.
(c) This is not a valid pdf because the function is negative for $0 < u < 1$.
However, $\int_0^1 \ln u \, du = u \ln u - u \Big|_0^1 = -1$, so $-\ln u, u \in (0,1)$ is a valid pdf.
(d) This is not a valid pdf because the function is negative for $u \in (0,1)$. Also, since the function is positive for $u \in (1,2)$, $Cf(u)$ cannot be a valid pdf for any C . (e) This is a valid pdf.
(f) Although $\int_0^3 (2/3)(u-1) \, du = (1/3)(u-1)^2 \Big|_0^3 = 1$, this is not a valid pdf because the function is negative for $u \in (0,1)$. Also, since the function is positive for $u \in (1,3)$, $Cf(u)$ cannot be a valid pdf for any C .
(g) Since $\int_0^\infty \exp(-2u) \, du = 1/2$, this is not a valid pdf. However, $2 \exp(-2u), u \in (0,\infty)$ is a valid pdf; it is the exponential density with parameter 2.
(h) Although $\int_0^\infty 4\exp(-2u) - \exp(-u) \, du = 2 - 1 = 1$, this is not a valid pdf because the function is negative for $u > \ln 4$. Since the function is positive for $u < \ln 4$, $Cf(u)$ cannot be a valid pdf for any C .
2. The pdf is as shown on the diagram below where some lines have been added to aid in computation. Each triangle shown has area $1/8$.



- (a) By inspection, we see that $P\{|X| < 1/2\} = 1/2$. Similarly, $P\{X < 1/2\} = 5/8$ and $P\{X > 0 \cap \{X < 1/2\}\} = P\{0 < X < 1/2\} = 1/8$, giving $P\{X > 0 \mid X < 1/2\} = 1/5$. Politically correct anti-segregationists (i.e. those who believe in integration) who failed to sketch the pdf can proceed as follows:
$$P\{|X| < 1/2\} = \int_{-1/2}^0 (1+u) \cdot du + \int_0^{1/2} u \cdot du = \int_{-1/2}^0 1 \cdot du + \int_{-1/2}^{1/2} u \cdot du = u \Big|_{-1/2}^0 + \frac{u^2}{2} \Big|_{-1/2}^{1/2} = \frac{1}{2} + 0 = \frac{1}{2}$$
 and so on.
(b)
$$E[X] = \int_{-\infty}^{\infty} u \cdot f_X(u) \cdot du = \int_{-1}^0 u \cdot (1+u) \cdot du + \int_0^1 u \cdot u \cdot du = \frac{u^2}{2} + \frac{u^3}{3} \Big|_{-1}^0 + \frac{u^3}{3} \Big|_0^1 = \frac{1}{6}.$$

(c)
$$E[|X|] = \int_{-\infty}^{\infty} |u| \cdot f_X(u) \cdot du = \int_{-1}^0 -u \cdot (1+u) \cdot du + \int_0^1 u \cdot u \cdot du = \frac{-u^2}{2} + \frac{-u^3}{3} \Big|_{-1}^0 + \frac{u^3}{3} \Big|_0^1 = \frac{1}{2}.$$
- 3.(a) No, the tank is empty after 500 gallons have been sold, and towards the end of the week, the people asking for the extra 180 gallons are s.o.l. (b) Sure, with 70 gallons to spare.
(c) The weekly demand can be satisfied if it (the demand) does not exceed 0.5. Thus,
$$P\{X \leq 0.5\} = \int_0^{0.5} 5(1-u)^4 \, du = -(1-u)^5 \Big|_0^{0.5} = 1 - \left(\frac{1}{2}\right)^5 = \frac{31}{32}$$

(d) We want the smallest value of C such that $P\{X > C\} \leq 10^{-5}$. But,
$$P\{X > C\} = \int_C^1 5(1-u)^4 \, du = -(1-u)^5 \Big|_C^1 = (1-C)^5 \leq 10^{-5} \text{ if } C \geq 0.9.$$
 Thus, a 900 gallon tank is required to achieve the desired goal.

$Y = \begin{cases} X & \text{if } X \leq C, \\ C & \text{if } X > C. \end{cases}$ Hence, the weekly gross profit is $640Y$, and the **average** weekly **gross** profit is

$$E[640Y] = \int_0^C 5(1-u)^4 640u \, du + \int_C^1 5(1-u)^4 640C \, du = \frac{640}{6}[1 - (1-C)^6].$$
 As a function of C , the size of the

tank, this increases from 0 if $C = 0$ (no tank!) to $\$640/6$ if $C = 1$. Now, the average **net** profit $= E[640Y - 20C] = E[640Y] - 20C = (640/6)[1 - (1-C)^6] - 20C$. This has a maximum if C satisfies $640(1-C)^5 = 20$ i.e. $(1-C)^5 = (1/32)$, i.e. if $C = 1/2$. Thus, a 500 gallon tank that costs $\$10$ per week to rent gives an average gross profit of $\$630/6$ and the maximum average net profit of $\$570/6$ per week. In contrast, a full-sized 1000 gallon tank gives only a slightly larger gross profit of $\$640/6$ per week, but a smaller net profit of $\$520/6$ per week. Of course, there may be intangible losses in goodwill if the tank is emptied before the week ends. However, the probability of this happening is only $1/32$ when $C = 1/2$.

$$4.(a) \quad E[Y] = E[X^2] = \int_{-1}^{+1} u^2(1/2)du = 1/3. \quad E[Y^2] = E[X^4] = \int_{-1}^{+1} u^4(1/2)du = 1/5.$$

$$\text{Hence, } \text{var}(Y) = E[Y^2] - (E[Y])^2 = (1/5) - (1/3)^2 = 4/45.$$

$$(b) \quad E[Z] = \int_{-\infty}^{\infty} g(u)f(u)du = \int_{-1}^{-\infty} -u^2(1/2)du + \int_0^1 u^2(1/2)du = -1/3 + 1/3 = 0.$$

5.(a) X is an exponential random variable with parameter λ ,

or X is a gamma random variable with parameters $(1, \lambda)$. In either case, $P\{0 < X \leq T\} = 1 - \exp(-\lambda T)$

(b) $P(A) = P\{\text{exactly one arrival in } (0, T]\} = P\{\text{Poisson RV with parameter } \lambda T \text{ has value } 1\} = \lambda T \exp(-\lambda T)$.

(c) The answers are different because the events are different. If $X \leq T$, it does not necessarily mean that there was exactly one arrival in $(0, T]$. However, if there was exactly one arrival in $(0, T]$, then we know for sure that $X \leq T$. Thus, the event A is a subset of the event $\{X \leq T\}$.

(d) $P\{X \leq \tau \mid A\} = P[\{X \leq \tau\} \cap A] / P(A) = P[\{1 \text{ arrival in } (0, \tau]\} \cap \{\text{no arrivals in } (\tau, T]\}] / P(A)$
 $= P\{1 \text{ arrival in } (0, \tau]\} \cdot P\{\text{no arrivals in } (\tau, T]\} / P(A)$ by independence of arrivals in disjoint intervals
 $= [\lambda \tau \exp(-\lambda \tau) \exp(-\lambda(T-\tau))] / \lambda T \exp(-\lambda T) = \tau/T$.

6.(a) The number of arrivals in the interval $(0, 4]$ is the Poisson random variable $N(0,4]$ with parameter 4λ . Hence, the mean number of arrivals is $E[N(0,4)] = 4\lambda$.

$$(b) \quad P[\{N(0, 3] = 3\} \cap \{N(2, 6] = 0\}] = P[\{N(0, 2] = 3\} \cap \{N(2, 6] = 0\}] = \left[\exp(-2\lambda) \frac{(2\lambda)^3}{3!} \right] \times \exp(-4\lambda)$$

$$= \exp(-6\lambda) \frac{(2\lambda)^3}{3!} \text{ since } N(0, 2] \text{ and } N(2, 6] \text{ are independent random variables } ((0,2] \text{ and } (2,6] \text{ are disjoint})$$

(c) The number of arrivals in $(0, 6]$ is the Poisson random variable $N(0,6]$ with parameter 6λ . The event $\{N(0,6] = 5\}$ has probability $\exp(-6\lambda) \frac{(6\lambda)^5}{5!}$, and this is maximized if $\lambda = 5/6$ as shown on a noncredit exercise in Problem Set #5.

$$(d) \quad \lambda = \ln 2. \quad P\{N(0, t] \geq 1\} = 1 - P\{N(0, t] = 0\} = 1 - \exp(-\lambda t) = 1 - 2^{-t}.$$