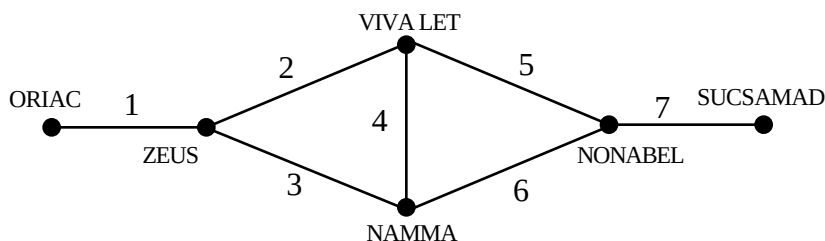


- 1.(a) With N-ary replication of each component, each replicated component has probability of failure p^N since all the N copies of the component must fail for the replicated component to fail. The probability that all M replicated components are in working order is thus $(1 - p^N)^M$, and the probability of failure for the system in the left-hand figure is thus $1 - (1 - p^N)^M$.
On the other hand, suppose that the entire system is replicated N times as shown in the right-hand figure. Since each copy of the entire system fails with probability $1 - (1 - p)^M$, the probability of failure for the system in the right-hand figure is $[1 - (1 - p)^M]^N$.
- (b) $1 - (1 - 0.2^N)^5 \leq 0.001 \quad (1 - 0.2^N)^5 \geq 0.999 \quad (1 - 0.2^N) \geq 0.99979992 \quad 0.2^N \leq 0.000200080$
 $N \geq 6$. More generally, $N \geq \frac{\ln(1 - (1 - \alpha)^{1/M})}{\ln p}$ where α is the desired maximum failure probability.
- $[1 - (1 - 0.2)^5]^N \leq 0.001 \quad 0.67232^N \leq 0.001 \quad N \geq 18$. More generally, $N \geq \frac{\ln \alpha}{\ln(1 - (1 - p)^M)}$
- Put another way, in our car example, having a single gigantic car with 6 engines, 6 transmissions, 6 brakes, ... will provide more reliable transportation than having 17 separate cars. This may (or may not!) explain why people buy one large SUV rather than 18 compact cars!
- (c) If $M = 1000$, we get $N \geq 9$ in the one case. The other case blows up on many calculators because the quantity $1 - (1 - p)^M \approx 1 - \exp(-Mp)$ (huh?) $= 1 - \exp(-200)$ is very nearly 1. The Unix high-precision utility *bc* gives its value as 0.999...99861610344 where there are 87 9's preceding the 8616.... Since most cars have 1000 (or more!) parts, the number of separate cars required to provide reliable transportation is far larger than the number of cars in this world! On the other hand, a car with nine engines, nine transmissions etc will provide the necessary reliability but it will be difficult to squeeze into most parking spaces!
- Moral:** Replicate at the component level, instead of replicating entire systems. It's a lot cheaper!

2. Linguistic innocents are reminded that both Arabic and Hebrew are read from right to left. Number the links as shown. Let V_i denote the event that the i-th link is viable (i.e. working) and let A denote the event that it is possible to call from ORIAC to SUCSAMAD.



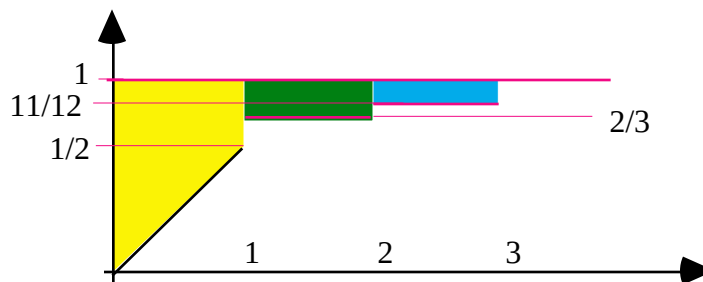
- (a) If V_4^c has occurred, i.e. the VIVA LET to NAMMA link has been severed, then,
 $P(A|V_4^c) = P(V_1 V_2 V_5 V_7 \mid V_1 V_3 V_6 V_7 | V_4^c) = P(V_1 V_2 V_5 V_7 \mid V_1 V_3 V_6 V_7)$ by independence!
 $= P(V_1 V_2 V_5 V_7) + P(V_1 V_3 V_6 V_7) - P(V_1 V_2 V_3 V_5 V_6 V_7) = 2(1 - p)^4 - (1 - p)^6$ by independence! On the other hand, if V_4 has occurred, i.e the VIVA LET to NAMMA link is viable, then
 $P(A|V_4) = P(V_1(V_2 \mid V_3)(V_5 \mid V_6)V_7|V_4) = P(V_1(V_2 \mid V_3)(V_5 \mid V_6)V_7)$ by independence!
 $= P(V_1) \times P(V_2 \mid V_3) \times P(V_5 \mid V_6) \times P(V_7) = (1 - p)^2 \times [(1 - p) + (1 - p) - (1 - p)^2]^2 = (1 - p)^2(1 - p^2)^2$
 $= (1 - p)^4(1 + p)^2$ by repeated use of the independence of events. Hence,
 $P(A) = p \times [2(1 - p)^4 - (1 - p)^6] + (1 - p) \times (1 - p)^4(1 + p)^2 = (1 - p)^4 \times (1 + 2p + p^2 - 2p^3)$
- (b) $P(V_3^c | A) = P(A \cap V_3^c) / P(A)$. But, $P(A \cap V_3^c) = P[V_1 V_2 (V_5 \mid V_4 V_6) V_7 V_3^c]$
 $= \{(1 - p)^2 \times [(1 - p) + (1 - p)^2 - (1 - p)^3] \times (1 - p) \times p = (1 - p)^4 \times (1 + p - p^2) \times p$. Hence,
 $P(V_3 | A) = 1 - P(V_3^c | A) = 1 - \frac{(1 - p)^4 \times (1 + p - p^2) \times p}{(1 - p)^4 \times (1 + 2p + p^2 - 2p^3)} = \frac{(1 + p - p^3)}{1 + 2p + p^2 - 2p^3}$.
- (c) Clearly X is nonnegative, and cannot exceed 70. Also, $X = 0$ if either link #1 or link #7 has been severed i.e. if $V_1^c \mid V_7^c$ occurs. Thus, assume that $V_1 V_7$ has occurred. As in part(a), we can calculate the capacity given that V_4^c has occurred and also given that V_4 has occurred. Thus, given that $V_1 V_4 V_7$ has occurred, the capacity from ORIAC to SUCSAMAD is 20 if only one of the paths from ZEUS to NONABEL is working, and 40 if both are working, etc.. These capacities are indicated on the two "Karnaugh" maps shown on the next page.

		$V_1 V_4 V_7$ occurred			
$V_2 V_5 \rightarrow$		00	01	11	10
$V_3 V_6 \downarrow$	00	0	0	20	0
	01	0	0	20	0
	11	20	20	40	20
	10	0	0	20	0

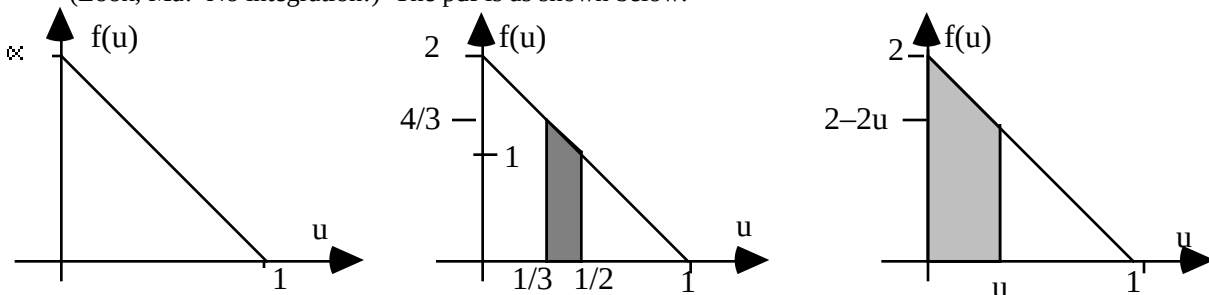
		$V_1 V_4 V_7$ occurred			
$V_2 V_3 \rightarrow$		00	01	11	10
$V_5 V_6 \downarrow$	00	0	0	0	0
	01	0	20	20	20
	11	0	50	70	20
	10	0	50	50	20

From this, we get $P\{X = 70\} = (1-p)^7$ {**check**: all links must work to get a capacity of 70},
 $P\{X = 50\} = [(1-p)^2 - (1-p)^4](1-p)^3 = (1-p)^5 - (1-p)^7 = (1-p)^5[2p - p^2]$ {**check**: links 1, 3, 4, 5, 7 must work, and at least one of 2 and 6 must have failed which has prob $p + p - p^2$, **why?**},
 $P\{X = 40\} = (1-p)^6 p$ {**check**: all links except 4 must be in working condition},
 $P\{X = 20\} = 2(1-p)^4 p^3 + 7(1-p)^5 p^2 + 2(1-p)^6 p$, {**check**: the first term corresponds to 1-2-5-7 and 1-3-6-7 as the only working links, the last term corresponds to 3 or 5 being the only failed link, and the middle term corresponds to everything working **except** (2 and exactly one of 4 and 5) or (3 and exactly one of 4, 5, 6) or (4 and exactly one of 5 and 6)},
 $P\{X = 0\} = 1 - P\{X \neq 0\} = 1 - [(1-p)^7 + (1-p)^5 - (1-p)^7 + p(1-p)^6 + 2(1-p)^4 p^3 + 7(1-p)^5 p^2 + 2(1-p)^6 p]$
 $= 1 - (1-p)^4(1 + 2p + p^2 - 2p^3) = 1 - P(A)$ as it should be!
 Finally, $E[X] = 70P\{X = 70\} + 50P\{X = 50\} + 40P\{X = 40\} + 20P\{X = 20\}$
 $= 20(1-p)^7 + 80(1-p)^6 p + 50(1-p)^5 + 40(1-p)^4 p^3 + 140(1-p)^5 p^2$. As a quick check, note that $E[X] = 70$ if $p = 0$ (totally incompetent terrorists!) and $E[X] = 0$ if $p = 1$ (fearless liberation fighters not afraid to die in the attempt!). It is interesting to plot these probabilities and expected values as a function of p . $P(A) \approx 80\%$ for $p = 0.05$ and decreases to 50% at roughly $p = 0.2419\dots$ when $E[X] = 21\dots$

3. Since X takes on positive values only, $E[X] = \int_0^{\infty} P\{X > u\} du$ = area between CDF and line at height 1 to the right of the origin $= (1 - (1/2) \cdot (1/2) \cdot 1) + 1/3 + 1/12 = 7/6$ (see the three areas shaded below).



4. (Look, Ma! No integration!) The pdf is as shown below.



- (a) $P\{6X^2 > 5X - 1\} = P\{6X^2 - 5X + 1 > 0\} = P\{(3X - 1)(2X - 1) > 0\}$
 $= P[\{(X > 1/3) \cap \{X > 1/2\}\} \cup \{(X < 1/3) \cap \{X < 1/2\}\}] = P\{X > 1/2\} \cup \{X < 1/3\}$
 $= 1 - P[1/3 \leq X \leq 1/2]$. Now, $P[1/3 \leq X \leq 1/2]$ = area under the pdf from 1/3 to 1/2.
 This is the difference of two triangular areas and is thus $(1/2) \times [(4/3) \times (2/3) - 1 \times (1/2)] = 7/36$.
 Hence, $P\{6X^2 > 5X - 1\} = 1 - P[1/3 \leq X \leq 1/2] = 1 - 7/36 = 29/36$.
- (b) $F_X(u)$ is the area under the pdf up to the point u . By inspection, we have that $F_X(u) = 0$ if $u < 0$ and $F_X(u) = 1$ if $u > 1$. For $0 \leq u \leq 1$, $F_X(u)$ is the area shaded in the right-hand figure above
 $= 1 - (1/2) \times (1-u) \times (2-2u) = 1 - (1-u)^2 = 2u - u^2$.
 The total area under the curve is $(1/2) \times \alpha \times 1 = \alpha/2$ which gives $\alpha = 2$.