

- 1.(a) For $x \in (0, \infty)$, $\ln x = \int_1^x \frac{1}{t} dt$ is a monotone increasing function of x (because the Fundamental Theorem of

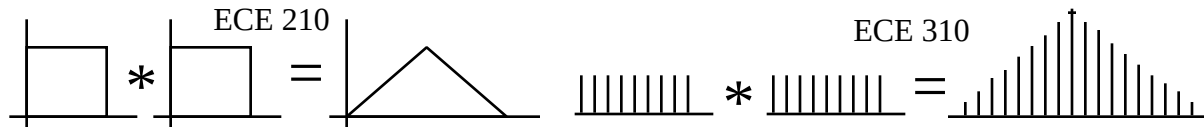
Calculus tells us that the derivative of $\ln x$ is $1/x$ which is positive for all $x \in (0, \infty)$).

Hence, $\ln e = 1 > 0 = \ln 1$, $e > 1$, $e^{-1} < 1$.

- (b) $\exp(-1) = \exp(-1^2) = \exp(0 - 1^2) = 0.3678\dots$ on my calculator
 (c) Excel calculates the values to be e^{-1} , e , and e^{-1} respectively. If you actually RTFM, you will find that according to Microsoft, the $-$ in $\text{EXP}(-1^2)$ is a *unary* operator that has priority over the exponentiation operator, that is, -1^2 means $(-1)^2$. On the other hand, the $-$ in $\text{EXP}(0-1^2)$ is an ordinary *binary* subtraction operator that, according to PEMDAS, has lower priority than the exponentiation operator.

Later on, when teaching you about Gaussian random variables, I will be writing $\exp(-x^2)$ a lot, so be sure to remember that I mean $\exp(-(x^2))$ and not $\exp((-x)^2) = \exp(x^2)$! Also, be very careful if and when you use Microsoft Excel in statistical applications!

- (d) Yes, ordinary multiplication by hand shows that $11111111^2 = 12345678987654321$. What does this have to do with the pictures shown below?



- (e) True: $\frac{1}{-\alpha \pm \sqrt{\beta}} = \frac{1}{-\alpha \pm \sqrt{\beta}} \cdot \frac{-\alpha \mp \sqrt{\beta}}{-\alpha \mp \sqrt{\beta}} = \frac{-\alpha \mp \sqrt{\beta}}{\alpha^2 - \beta^2}$ showing that $\frac{1}{-b \pm \sqrt{b^2 - 4ac}} = \frac{-b \mp \sqrt{b^2 - 4ac}}{4ac}$

It follows that the given expression simplifies to the more familiar $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- 2.(a) You were not asked to provide a *proof* that the answer to part (a) is 3, but here is one for anyone interested. $\cos(a-b) + \cos(a+b) = 2 \cos a \cos b$, while $\cos(a-b) - \cos(a+b) = 2 \sin a \sin b$.

Also, $\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$, and $\cos 3x = 4 \cos^3 x - 3 \cos x$.

$$\text{Hence, } \cot(10)\cot(30)\cot(50)\cot(70) = \frac{(\cos 80 + \cos 60)(\cos 80 + \cos 20)}{(\cos 80 - \cos 60)(\cos 80 - \cos 20)}.$$

Let $\cos 20 = y$. Since $\cos 60 = 1/2$, we know that $4y^3 - 3y = 1/2$, and that $\cos 80 = 8y^4 - 8y^2 + 1$

$= 2y(4y^3 - 3y) - 2y^2 + 1 = 1 + y - 2y^2$. Thus, the above fraction equals

$$\frac{(3/2 + y - 2y^2)(1 + 2y - 2y^2)}{(1/2 + y - 2y^2)(1 - 2y^2)} = \frac{3/2 + 4y - 3y^2 - 6y^3 + 4y^4}{1/2 + y - 3y^2 - 2y^3 + 4y^4} = \frac{3/4}{1/4} = 3$$

on substituting $y^3 = (3/4)y + 1/8$ and $4y^4 = 3y^2 + y/2$.

- (b) For any x , $0 \leq x < 90$, $\cot(x) = \tan(90-x)$ and $\cot(x) = 1/\tan(x)$. Hence, $\tan(80)\tan(60)\tan(40)\tan(20) = 3$, and, on replacing all the tans by 1/cots, we deduce that $\cot(20)\cot(40)\cot(60)\cot(80) = 1/3$. Note that it is easy to show that $\cot(45-y)\cot(45-x)\cot(45+x)\cot(45+y) = 1$. It is not quite as easy to find such formulas if the RHS is required to be an integer other than 1 (as in the surprising result of part (a)).

- (c) Since $\frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \cdot \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}} = \frac{\sqrt{n+1} - \sqrt{n}}{(n+1) - (n)} = \sqrt{n+1} - \sqrt{n}$, the sum “telescopes”

to $\sqrt{100} - \sqrt{1} = 9$. Look, Ma! No calculator.

- 3.(a) 7. (b) My cheap calculator gives 0 as the answer in all cases. This seems reasonable because $\sin x \approx x$ for small x , and thus the function should have value nearly 0 as x approaches 0. However, the answers that you get *will* depend on the number of digits of precision used by your calculating device as well as the algorithm it uses to compute $\sin(x)$. For example, at $x = 10^{-7}$, the Unix high-precision calculator utility *bc* gives the following values: $\approx 10^9$ using 20 digits; $\approx 10^3$ using 25 digits, ≈ 0.34 using 30 digits and ≈ 0.33333333333334 using 45 or more digits. Thus, it would appear that the limit might be $1/3$. For the formal proof, note $\sin x = x - x^3/3! + x^5/5! - \dots \approx x - x^3/6$ for small x . Hence, for small x ,

$$\frac{1}{[\sin x]^2} - \frac{1}{x^2} \approx \frac{1}{x^2} \frac{1}{(1 - x^2/6)^2} - \frac{1}{x^2} = \frac{1}{x^2} \left[1 + 2x^2/6 + 3(x^2/6)^2 + \dots - 1 \right] = \frac{1}{3} + \frac{3x^2}{36} \dots$$

Here we have used the series: $(1-y)^{-2} = 1 + 2y + 3y^2 + \dots$ which is valid for $|y| < 1$.

Remember this series; it and the series $(1-y)^{-1} = 1 + y + y^2 + y^3 + \dots$ will be used many times in the course.

Exercise: What do you get when you differentiate both sides of $(1-y)^{-1} = 1 + y + y^2 + y^3 + \dots$

Thus, $\frac{1}{[\sin x]^2} - \frac{1}{x^2} \approx 1/3 + 3x^2/36 + \dots = 1/3$ at $x = 0$, i.e., $\lim_{x \rightarrow 0} \frac{1}{[\sin x]^2} - \frac{1}{x^2} = \frac{1}{3}$

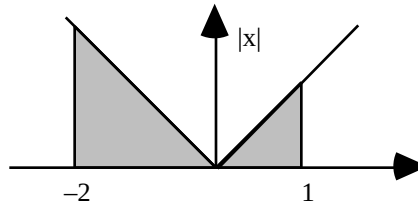
Those who prefer a purer approach should note that $\frac{1}{[\sin x]^2} - \frac{1}{x^2} = \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$ is of the form $0/0$ at $x = 0$.

Thus, by (repeated application of) L'Hôpital's rule and the identity $\frac{d}{dx} \sin^2 x = \sin(2x)$ (huh??), we have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1}{[\sin x]^2} - \frac{1}{x^2} &= \lim_{x \rightarrow 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x \rightarrow 0} \frac{2x - \sin(2x)}{2x \sin^2 x + x^2 \sin(2x)} \\ &= \lim_{x \rightarrow 0} \frac{2 - 2\cos(2x)}{2\sin^2 x + 4x \sin(2x) + 2x^2 \cos(2x)} = \lim_{x \rightarrow 0} \frac{4\sin(2x)}{2\sin(2x) + 12x \cos(2x) - 4x^2 \sin(2x) + 4\sin(2x)} \\ &= \lim_{x \rightarrow 0} \frac{8\cos(2x)}{4\cos(2x) + 12\cos(2x) + 8\cos(2x) + x(\text{some other stuff})} = \frac{8}{24} = \frac{1}{3} \text{ just as before.} \end{aligned}$$

(c) $x^n \exp(-ax)$ has maximum value $(n/a)^n \exp(-n)$ at $x = n/a$. Here $n = 25$, $a = \ln 1.00001$ giving a maximum value of $0.123365 \dots \times 10^{150}$ at $x = 2,500,012.5 \dots$

4.(a) From the picture: $\int_{-2}^1 |x| dx$ = the shaded area = sum of the areas of two triangles
(area = $(1/2) \times \text{base} \times \text{altitude}$, remember?) = $(1/2) \times 2 \times 2 + (1/2) \times 1 \times 1 = 2.5$.



Those who insist on integration should note that $|x| = x$ if $x > 0$ and $|x| = -x$ if $x < 0$, and hence $\int_{-2}^1 |x| dx =$

$$\int_{-2}^0 -x dx + \int_0^1 x dx = \left. -\frac{x^2}{2} \right|_{-2}^0 + \left. \frac{x^2}{2} \right|_0^1 = 2.5. \text{ The answer is NOT } \int_{-2}^1 |x| dx = \left. \frac{|x|^2}{2} \right|_{-2}^1 = \frac{1^2}{2} - \frac{(-2)^2}{2} = -1.5$$

It is very important to learn the habit of drawing a sketch before tackling integrals in this course. It will save you from a lot of errors, and in some cases, it will ease your task considerably. The above is an example of this.

Put $y = 1 - x$. Then, $\int_{-2}^1 x(1-x)^{19} dx = \int_3^0 (1-y)y^{19} (-dy) = \int_0^3 y^{19} - y^{20} dy = \frac{3^{20}}{20} - \frac{3^{21}}{21} = \frac{-13 \times 3^{20}}{140}$

= -323,772,837.2.... Alternatively, integration by parts can be used. Did you get a positive answer?

(b) Since an integral is "the area under the curve" the integral of a nonnegative function cannot be negative. A proof of this can be written using (for example) the comparison property (see, e.g. Edwards and Penney).

I hope that warning bells will go off in your mind if you ever find the integral of a positive function to be a negative number. These bells will peel out more strongly if your sixth sense is aided by a sketch indicating just which area corresponds to the integral that you are computing.

(c) (i) False: the chain rule gives $-g(-x)$ as the derivative. (ii) True as per the chain rule. (iii) False: the chain rule gives $\exp(f(x^2))g(x^2)2x$ as the derivative (iv) True. cf. (i) (v) False: the antiderivative of $g(x^2)$ need not be related to $f(x^2)$ at all! (vi) True only for positive functions $f(x)$.

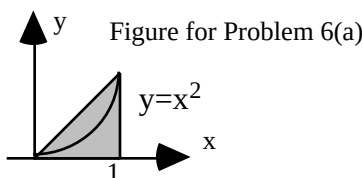
(d) Since $\frac{d}{dx} \exp(-x^2/2) = -x \exp(-x^2/2)$, $\int_0^\infty x \exp(-x^2/2) dx = -\exp(-x^2/2) \Big|_0^\infty = 1$.

5.(a) $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$. (b) $I = \int_{-1}^1 \frac{2}{1+x^2} dx = 2 \cdot \arctan(x) \Big|_{-1}^1 = 2 \cdot [\pi/4 - (-\pi/4)] = \pi$.

- (c) True. The value of the integral does not depend on the name of the variable of integration.
- (d)(e) The substitution $y = 1/x$, $dx = -y^{-2} dy$, $x = -1 \quad y = -1$, $x = 1 \quad y = 1$ only **seemingly** leads to the result that $I = -J$. In fact, although it is perfectly true that $x = -1 \quad y = -1$, it so happens that as x **increases** from -1 to 0 , y **decreases** from -1 to $-\infty$. Thus, the change of variables really gives

$$\begin{aligned}
 I &= \int_{-1}^1 \frac{2}{1+x^2} dx = \int_{-1}^0 \frac{2}{1+x^2} dx + \int_0^1 \frac{2}{1+x^2} dx = \int_{-1}^{-\infty} \frac{-2}{1+y^2} dy + \int_{\infty}^1 \frac{-2}{1+y^2} dy = \int_{-\infty}^{-1} \frac{2}{1+y^2} dy + \int_1^{\infty} \frac{2}{1+y^2} dy \\
 &= 2 \cdot \arctan(x) \Big|_{-\infty}^{-1} + 2 \cdot \arctan(x) \Big|_1^{\infty} = 2 \cdot [(-\pi/4) - (-\pi/2)] + 2[(\pi/2) - (\pi/4)] = \pi \text{ just as before.}
 \end{aligned}$$

Sorry, folks, and don't forget to re-change all those 0's back to π 's in your textbooks!



- 6.(a) $f(x,y)$ is nonzero only on the triangular shaded region shown above. Hence,

$$\int_{y > x^2} \int f(x,y) dx dy = \int_{x=0}^1 \int_{y=x^2}^x 6 dy dx = \int_{x=0}^1 6y \Big|_{x^2}^x dx = \int_{x=0}^1 6x - 6x^2 dx = 3x^2 - 2x^3 \Big|_0^1 = 1.$$

Alternatively, the area under the parabola x^2 from 0 to a is $a^3/3$. Thus, the area between the parabola and the hypotenuse of the triangle is $1/6$. Since $f(x,y) = 6$ on that region, the integral = volume between $f(x,y)$ and x - y plane = $6(1/6) = 1$.

- (b) Changing to polar coordinates $\int \int (x^2 + y^2)^{-2} dx dy = \int_{\sqrt{2}}^{\infty} \int_0^{2\pi} r^{-4} r d\theta dr = 2\pi \int_{\sqrt{2}}^{\infty} r^{-3} dr = \pi/2$.