University of Illinois	**Solutions to First Hour Exam**	**ECE 313 Spring 2002**

1. A wins if the first ball drawn is red, or if the first and second balls are black and the third is red. The probability is thus $\frac{1}{2} + \left[\frac{3}{6} \times \frac{2}{5} \times \frac{3}{4}\right] = \frac{1}{2} + \frac{3}{20} = \frac{13}{20}$. OR, just list all 20 outcomes and count!

2.(a) P(A B) = P(A) + P(B) – P(A ∩ B). Hence, conditioning on the occurrence of A B, we have that P(A B|A B) = P(A|A B) + P(B|A B) – P(A ∩ B|A B). But, obviously, P(A B|A B) = 1 and thus we get that P(A ∩ B|A B) = $\frac{3}{5} + \frac{4}{5} - 1 = \frac{2}{5}$. Similarly, conditioning the result that P(A B) = P(A) + P(B) – 2×P(A ∩ B) on the occurrence of A B, we get that

P(A B|A B) = P(A|A B) + P(B|A B) – 2×P(A ∩ B|A B) = $\frac{3}{5} + \frac{4}{5} - 2 \times \frac{2}{5} = \frac{3}{5}$.

Finally, since A B A B, P(A B|A B) = 1.

(b) FALSE:. The conditioning needs to be the other way around, as we used in part (a).

TRUE: From Bayes' formula, we have that P(C|D) = P(D|C)P(C)/P(D) = P(D|C) if P(C) = P(D).

FALSE!!! If P(DE) = 0, then P(E|D) = P(D|E) will hold even if P(D) ≠ P(E).

3.(a) There are $\binom{5}{2} = \frac{5 \times 4}{1 \times 2} = 10$ possible pairs of letters, consisting of the pair {I, I}, 6 pairs of the form {I, X} where X is one of {M, M, A}, 2 pairs {M, A}, and the pair {M, M}. The conditional probability that the sign still seems to read MIAMI is 1, 1/4, 1/8, and 1/4 respectively for these pairs falling down. The theorem of total probability thus gives

P{sign still seems to read MIAMI} = $\left[1 \times \frac{1}{10} + \frac{1}{4} \times \frac{6}{10} + \frac{1}{8} \times \frac{2}{10} + \frac{1}{4} \times \frac{1}{10}\right] = \frac{12}{40} = \frac{3}{10}$.

(b) From Bayes' formula, we get that P{2 M's fell down |sign still seems to read MIAMI}

$= \frac{P\{\text{sign still seems to read MIAMI } | \text{2 M's fell down}\}P\{\text{2 M's fell down}\}}{P\{\text{sign still seems to read MIAMI}\}} = \frac{1/40}{12/40} = \frac{1}{12}$.

4.(a) The likelihood matrix and maximum-likelihood decision rule is as shown below.

H_0: student is from Section A	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{1}{6}$
H_1: student is from Section D	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{3}{10}$
Observed classification →	**Excellent**	**Good**	**Average**

Sanity check: Each row adds up to 1, just as it should for a likelihood matrix. The desired conditional probabilities can be read off from the likelihood matrix as shaded.

P{student from Section A is assumed to be in Section D} = P{average student from Section A} = $\frac{1}{6}$.

P{student from Section D is assumed to be in Section A} = P{above average student from Section D} = $\frac{7}{10}$.

Since the probability that the student is in Section A is $\frac{3}{8}$, we get P{incorrect decision} = $\frac{1}{6}\times\frac{3}{8} + \frac{7}{10}\times\frac{5}{8} = \frac{1}{2}$.

More simply, 5 students from Section A and 35 students from Section D, that is, a total of 40 out of the 80 students, are mis-classified.

(c) The joint probability matrix and minimum-error-probability decision rule are as shown.

H_0: student is from Section A	$\frac{2}{6}\times\frac{3}{8} = \frac{2}{16}$	$\frac{3}{6}\times\frac{3}{8} = \frac{3}{16}$	$\frac{1}{6}\times\frac{3}{8} = \frac{1}{16}$
H_1: student is from Section D	$\frac{3}{10}\times\frac{5}{8} = \frac{3}{16}$	$\frac{4}{10}\times\frac{5}{8} = \frac{4}{16}$	$\frac{3}{10}\times\frac{5}{8} = \frac{3}{16}$
Observed classification →	**Excellent**	**Good**	**Average**

Sanity check: The sum of all the entries is 1. Note that we always decide the student is from Section D!

(d) P{incorrect decision} = P(chosen student is from Section A} = $\frac{3}{8} < \frac{1}{2}$, the answer in part (b). It better be!

5.(a) $E[\mathbf{X}] = 1/p = 3$. $\text{var}(\mathbf{X}) = E[(\mathbf{X} - 3)^2] = (1-p)/p^2 = (2/3)/(1/9) = 6$ $E[(\mathbf{X} - 2)^2] = \text{var}(\mathbf{X}) + (2-3)^2 = 7$.

More long-windedly, $E[(\mathbf{X} - 2)^2] = E[\mathbf{X}^2] - 4\bullet E[\mathbf{X}] + 4 = \text{var}(\mathbf{X}) + 3^2 - 4\bullet E[\mathbf{X}] + 4 = 7$.

(b) $P\{\mathbf{X} = 2 \mid \mathbf{X} < 4\} = \frac{P\{\mathbf{X} = 2\}}{P\{\mathbf{X} < 4\}} = \frac{P\{\mathbf{X} = 2\}}{1 - P\{\mathbf{X} > 3\}} = \frac{(2/3)(1/3)}{1 - (2/3)^3} = \frac{2/9}{1 - 8/27} = \frac{6/27}{19/27} = \frac{6}{19}$.