ECE 313
FINAL EXAMINATION
Saturday May 6, 2000
3 hours

1. (15 points) Check the appropriate box in each part. No justification is required. Scoring will be as follows: You get +5 points for the right answer; 0 for no answer, and –1 point for a wrong answer.

(a) Let A and B denote events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.
If $P(A|B) = P(B|A)$, which of the following four statements are true statements?

☐ ‹ A and B are independent events
☐ ` A and B are disjoint events
☐ ´ $P(A) = P(B)$
☐ ^ $P(A^c|B^c) = P(B^c|A^c)$

☐ *Only* ‹ is a true statement
☐ *Only* ´ is a true statement
☐ *Only* ^ is a true statement
☐ *Exactly one* of ` and ´ is a true statement
☐ *Both* ` and ´ are true statements
☐ None of the above five choices correctly describes which are the true statements

(b) Let **X** denote a *continuous* random variable with probability density function $f_{\mathbf{X}}(u)$ and cumulative probability distribution function $F_{\mathbf{X}}(u)$. Which of the following four statements are true statements?

☐ ‹ $f_{\mathbf{X}}(u) \le 1$ for all u, $-\infty < u < \infty$.
☐ ` $\lim_{u \to -\infty} f_{\mathbf{X}}(u) = 0$
☐ ´ $\lim_{u \to \infty} f_{\mathbf{X}}(u) = 1$
☐ ^ $\lim_{u \to \infty} F_{\mathbf{X}}(u) = 1$

☐ *Only* ‹ , ` , and ^ are true statements
☐ *Only* ‹ and ^ are true statements
☐ *Only* ` and ^ are true statements
☐ *Only* ` , ´ and ^ are true statements
☐ *All four* are true statements
☐ None of the above five choices correctly describes which are the true statements

(c) Let **X** and **Y** denote jointly continuous random variables and suppose that $P\{\mathbf{X} > \mathbf{Y}\} = P\{\mathbf{X} < \mathbf{Y}\} = 1/2$. Which of the following four statements are true statements?

☐ ‹ **X** and **Y** are independent
☐ ` **X** and **Y** have identical marginal pdfs
☐ ´ $E[\mathbf{X}] = E[\mathbf{Y}]$
☐ ^ $P\{\mathbf{X} > 2\mathbf{Y}\} = P\{\mathbf{X} < \mathbf{Y}/2\}$

☐ *Only* ‹ is a true statement
☐ *Only* ‹ and ` are true statements
☐ *Only* ´ is a true statement
☐ *All four* are true statements
☐ *All four* are **false** statements
☐ None of the above five choices correctly describes which are the true statements

Note: for jointly continuous random variables, $P\{\mathbf{X} = \mathbf{Y}\} = 0$ since the "curve" $v = u$ has no area…

2. (84 points) Let $\mathbf{X}$ denote a *continuous* random variable with pdf $f_{\mathbf{X}}(u)$ which is an even function of u, i.e., $f_{\mathbf{X}}(u) = f_{\mathbf{X}}(-u)$ for all u, $-\infty < u < \infty$.

Let $F_{\mathbf{X}}(u)$ denote the CDF of $\mathbf{X}$, and suppose that $\text{var}(\mathbf{X}) = 9$.

Let $\mathbf{Y} = |\mathbf{X}|$ and $\mathbf{Z} = -\mathbf{X}$, let $f_{\mathbf{Y}}(v)$ and $f_{\mathbf{Z}}(w)$ denote their respective pdfs, and $F_{\mathbf{Y}}(v)$ and $F_{\mathbf{Z}}(w)$ denote their respective CDFs.

For each of the statements below,
mark **YES** if the statement **is true for all pdfs** satisfying the above conditions;
mark **NO** if the statement **is false for all such pdfs;**
and mark **MAYBE** if the statement is **true for some pdfs** and **false for others**.

No justification is required. Scoring will be as follows: +4 points for the right answer; 0 for no answer, and –2 points for a wrong answer.

YES	NO	MAYBE	
☐	☐	☐	$F_{\mathbf{X}}(u) = F_{\mathbf{X}}(-u)$ for all u, $-\infty < u < \infty$.
☐	☐	☐	$P\{\mathbf{X} > \alpha\} = F_{\mathbf{X}}(-\alpha)$ for all α, $-\infty < \alpha < \infty$.
☐	☐	☐	$F_{\mathbf{Y}}(v) = F_{\mathbf{X}}(\lvert v\rvert)$ for all v, $-\infty < v < \infty$.
☐	☐	☐	$f_{\mathbf{Y}}(v) = \begin{cases} 2f_{\mathbf{X}}(v) & \text{if } v \geq 0, \\ 0 & \text{if } v < 0. \end{cases}$
☐	☐	☐	$F_{\mathbf{Z}}(w) = F_{\mathbf{X}}(-w)$ for all w, $-\infty < w < \infty$.
☐	☐	☐	$f_{\mathbf{Z}}(w) = f_{\mathbf{X}}(-w)$ for all w, $-\infty < w < \infty$.
☐	☐	☐	$f_{\mathbf{Z}}(w) = -f_{\mathbf{X}}(-w)$ for all w, $-\infty < w < \infty$.
☐	☐	☐	$E[\mathbf{X}] = 0$
☐	☐	☐	$E[\mathbf{Y}] = 3$
☐	☐	☐	$E[\mathbf{Y}^2] = 9$
☐	☐	☐	$\text{var}(\mathbf{Y}) < 8$
☐	☐	☐	$\text{var}(\mathbf{Y}) < 9$
☐	☐	☐	$E[\mathbf{XY}] = 0$
☐	☐	☐	$\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated random variables
☐	☐	☐	$\mathbf{X}$ and $\mathbf{Y}$ are independent random variables
☐	☐	☐	$E[\mathbf{XZ}] = -9$
☐	☐	☐	$P\{\mathbf{X} + \mathbf{Y} \leq 0\} = 1/2$
☐	☐	☐	$F_{\mathbf{Y}}(6) \geq 3/4$
☐	☐	☐	$F_{\mathbf{X}}(6) \geq 0.9772$
☐	☐	☐	$P\{\mathbf{X}^2 + 4\mathbf{X} + 3 < 0\} = F_{\mathbf{X}}(-1) - F_{\mathbf{X}}(-3)$
☐	☐	☐	$P\{\mathbf{X}^2 + 4\mathbf{X} + 3 > 0\} = P\{\mathbf{X}^2 - 4\mathbf{X} + 3 > 0\}$

3. **(30 points)** A, B, and C are three events such that P(A) = P(B) = P(C) = 1/2. The probability that *at least one* of the three events occurred is 0.8, while the probability that *at least two* of the three events occurred is 0.6.

(a) **(10 points)** What is the probability that all three events occurred ?

(b) **(10 points)** Given that at least two of the three events occurred, what is the conditional probability that all three occurred ?

(c) **(10 points)** Given that A occurred, what is the conditional probability that at least two of the events occurred ?

Warning: **Do not** assume that A, B, and C are independent events. If an answer cannot be determined from the given data, check the box on the left.

4. **(20 points)** The Gaussian random variable **X** has mean 3 and variance 100. Find the probability that the square root of the magnitude of 1 + **X** exceeds 4**.** A numerical answer correct to four decimal places is desired.

5. **(15 points)** **X** and **Y** are Bernoulli random variables with parameters $r_{\mathbf{X}}$ and $r_{\mathbf{Y}}$ respectively. It is known that

$$p_{\mathbf{X},\mathbf{Y}}(0,1) = P\{\mathbf{X} = 0, \mathbf{Y} = 1\} = 0.2 \text{ and } p_{\mathbf{X},\mathbf{Y}}(1,0) = P\{\mathbf{X} = 1, \mathbf{Y} = 0\} = 0.3.$$

You are to determine whether or when **X** and **Y** are independent random variables. Check one of the boxes below and fill in the blank if appropriate. **SHOW YOUR WORK!**

- ☐ **X** and **Y** are independent random variables if and only if $p_{\mathbf{X},\mathbf{Y}}(1,1)$ has the following value(s) _____ (if you need more space than this, you are in trouble!)
- ☐ **X** and **Y** are not independent random variables no matter what the value of $p_{\mathbf{X},\mathbf{Y}}(1,1)$ happens to be (in the range $0 \le p_{\mathbf{X},\mathbf{Y}}(1,1) \le 0.5$, of course).
- ☐ The information provided is insufficient to draw any conclusions as to whether or when **X** and **Y** can be independent random variables.

6. **(31 points)** **X** and **Y** are jointly continuous random variables with joint probability density function (pdf) given by $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} \exp(-u), & 0 < v < u < \infty \\ 0, & \text{elsewhere.} \end{cases}$

(a) **(10 points)** What are the values of $P\{\mathbf{Y} < \mathbf{X}\}$ and $P\{2\mathbf{Y} < \mathbf{X}\}$?

(b) **(14 points)** Let $\mathbf{Z} = \mathbf{Y}/\mathbf{X}$. Find $P\{\mathbf{Z} \le \alpha\}$ where α is some fixed number in the range $0 < \alpha < 1$.

(c) **(7 points)** Find $f_{\mathbf{Z}}(\alpha)$, the probability density function of **Z**. To obtain full credit, you must specify the value of $f_{\mathbf{Z}}(\alpha)$ for all α, $-\infty < \alpha < \infty$.

7. **(30 points)** **X** and **Y** are independent exponential random variables with parameter 1. Find the mean and variance of $\mathbf{Z} = |\mathbf{X} - \mathbf{Y}|$.