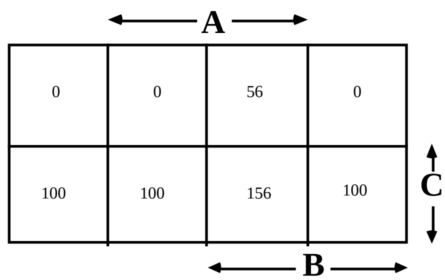
University of Illinois | **Solutions to Second Hour Exam** | **ECE 313 Spring 2000**

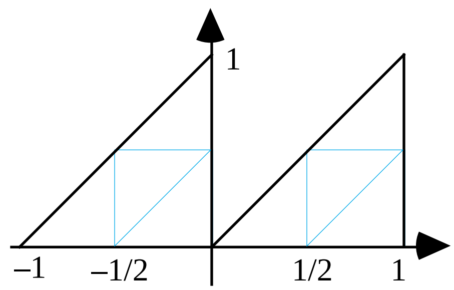
1. P{**X** is even} = P{**X** = 2} + P{**X** = 4} + P{**X** = 6} + … = $qp + q^3p + q^5p + \ldots = qp[1 + q^2 + q^4 + \ldots]$ $= \frac{q(1-q)}{1-q^2} = \frac{q}{q+1} = \frac{1}{6}$ which gives $q = \frac{1}{5}$ and $p = \frac{4}{5}$.

Hence, P{**X** is a multiple of 3} = P{**X** = 3} + P{**X** = 6} + P{**X** = 9} + … = $q^2p + q^5p + q^8p + \ldots$ $= q^2p[1 + q^3 + q^6 + \ldots] = \frac{q^2(1-q)}{1-q^3} = \frac{q^2}{q^2+q+1} = \frac{1}{31}$.

2. Let A, B, and C respectively denote the events that the U-O, O-C, and U-C links are in working condition. The capacities are marked on the Karnaugh map below (in which each cell has probability 1/8). It is easily seen that **Z** takes on values 0, 56, 100, and 156 with probabilities 3/8, 1/8, 3/8, and 1/8 respectively, and hence E[**Z**] = $0\times\frac{3}{8} + 56\times\frac{1}{8} + 100\times\frac{3}{8} + 156\times\frac{1}{8} = \frac{0 + 56 + 300 + 156}{8} = \frac{512}{8} = 64$.



3. The pdf is as shown on the diagram below where some lines have been added to aid in computation. Each triangle shown has area 1/8.



(a) By inspection, we see that P{|**X**| < 1/2} = 1/2. Similarly, P{**X** < 1/2} = 5/8} and P[{**X** > 0} ∩ {**X** < 1/2} = P{0 < **X** < 1/2} = 1/8, giving P{**X** > 0 | **X** < 1/2} = 1/5. Politically correct anti-segregationists (i.e. those who believe in integration) who failed to sketch the pdf can proceed as follows:

P{|**X**| < 1/2} = $\int_{-1/2}^{0}(1+u)\cdot du + \int_{0}^{1/2} u\cdot du = \int_{-1/2}^{0} 1\cdot du + \int_{-1/2}^{1/2} u\cdot du = u\Big|_{-1/2}^{0} + \frac{u^2}{2}\Big|_{-1/2}^{1/2} = \frac{1}{2} + 0 = \frac{1}{2}$ and so on.

(b) E[**X**] = $\int_{-\infty}^{\infty} u\cdot f_X(u)\cdot du = \int_{-1}^{0} u\cdot(1+u)\cdot du + \int_{0}^{1} u\cdot u\cdot du = \frac{u^2}{2} + \frac{u^3}{3}\Big|_{-1}^{0} + \frac{u^3}{3}\Big|_{0}^{1} = \frac{1}{6}$.

(c) E[|**X**|] = $\int_{-\infty}^{\infty} |u|\cdot f_X(u)\cdot du = \int_{-1}^{0} -u\cdot(1+u)\cdot du + \int_{0}^{1} u\cdot u\cdot du = \frac{-u^2}{2} + \frac{-u^3}{3}\Big|_{-1}^{0} + \frac{u^3}{3}\Big|_{0}^{1} = \frac{1}{2}$.

(d) $g(u) = |u|$. Hence, for any $v > 0$, we have two solutions $u_1 = v$ and $u_2 = -v$ to the equation $g(u) = v$. Since the absolute value of the derivative of g(u) is 1 at both points, we get $f_Y(v) = f_X(v) + f_X(-v)$ for $v > 0$. Obviously $f_X(v) = f_X(-v) = 0$ if $|v| > 1$, and hence we have that $f_Y(v) = v + (1-v) = 1$ for $0 < v < 1$. In summary, $f_Y(v) = \begin{cases} 1, & 0 < v < 1, \\ 0, & \text{otherwise,} \end{cases}$ which is readily seen to be a valid pdf.

(e) Since **Y** is uniformly distributed on (0,1), its mean value is 1/2 (as you have written down on your sheet of notes, I hope!). Otherwise, compute E[**Y**] = $\int_{0}^{1} v\cdot 1\cdot dv = \frac{v^2}{2}\Big|_{0}^{1} = \frac{1}{2}$ to get the same answer as in part (c).