

Assigned: Wednesday, March 24, 1999

Due: Wednesday, March 31, 1999

Reading: Ross, Chapter 5

Noncredit Exercises: Ross, pp. 232-237: 5-9, 12, 13, 15-19, 21, 26, 29, 30-34;
pp. 237-241: 2, 3, 8, 12-22, 26, 29; pp. 241-243: 1-4, 9, 10, 13-16.

Problems:

1. Participants in a lottery buy tickets for \$1 each. Each ticket bears one of the numbers 1 to 32. One of these 32 numbers is drawn at random, and all holders of tickets with the winning number are paid \$29 per ticket (i.e. the ticket price plus winnings of \$28).
 - (a) If you buy a ticket numbered 7, what is the probability that you win? If you buy a ticket for each drawing of the lottery, what is the **average** number of drawings until you win? Is this average number calculation valid only if you buy a ticket numbered 7 each time, or can you vary your choice with each drawing?
 - (b) What is your average win (or loss) per drawing?
 - (c) If you buy 32 tickets numbered 1 through 32, you will always hold a winning ticket. But, how much money do you lose on each drawing? What is your loss **per ticket**?
 - (d) In response to complaints that the lottery payoffs are too low, the administrator claims that *on average*, number 7 (say) wins at least once in 22 drawings while he is paying out \$28 on winning tickets. Thus, the lottery is actually biased in the player's favor! He backs up his claim by offering the following even-money bet: if 7 does not **win** in the next 22 drawings, he will pay you \$100, while if 7 does win, then you will pay him \$100. You accept this wager and sure enough, 7 wins on the twelfth drawing. "Pay up" says the administrator, and after collecting, asks if you want to bet that 7 does not win the *next* 22 drawings. You accept, and this time, 7 does not win in the next 22 drawings and you get your money back. You play this game with the administrator many times, and find that he is right! On roughly half of the bets, 7 does not occur in the next 22 drawings and you collect, but on the other half of your bets, 7 does occur sooner, and you have to pay up. But, is the bet *perfectly* fair (in the sense that the expected win/loss per bet is 0)? Does it prove the administrator's claim that 7 wins at least once (on average) in 22 drawings? If you agree with the administrator, how do you reconcile this with part (a) where you found (I hope!) that 7 wins once in 32 drawings (on average?)
2. X is uniformly distributed on $[-1, +1]$.
 - (a) If $Y = X^2$, what are the mean and variance of Y ?
 - (b) If $Z = g(X)$ where $g(u) = \begin{cases} u^2, & u \geq 0, \\ -u^2, & u < 0, \end{cases}$ use LOTUS (or the EZ method) to find $E[Z]$
3. We return to Problem 6 of Problem Set #8. Suppose that the owner makes a gross profit of \$0.64 for each gallon of gasoline sold. Let Y denote the amount of gasoline *sold* per week.
 - (a) How is Y related to X , the weekly *demand* for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!)
 - (b) What is the **average** weekly gross profit?
 - (c) Suppose that the owner pays \$20C as weekly rent on a tank of capacity 1000C gallons. Note that $0 \leq C \leq 1$. (Why is a tank larger than 1000 gallons not needed?) What is the **average** weekly **net** profit and what value of C maximizes the average weekly net profit?
4. X is a geometric random variable with parameter $1/2$, and $Y = \sin(\pi X/2)$. Is Y a continuous random variable or a discrete random variable or a mixed random variable?
If you think that Y is a continuous random variable, find the pdf of Y .
If you think that Y is a discrete random variable, find the pmf of Y .
If you think that Y is a mixed random variable, find the CDF of Y .

5. The random variable $\mathbf{X}$ has probability density function $f_{\mathbf{X}}(u) = \begin{cases} 2(1-u), & 0 \leq u \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$
Let $\mathbf{Y} = (1 - \mathbf{X})^2$.
- (a) What is the minimum value of $\mathbf{Y}$? Call this α .
What is the maximum value of $\mathbf{Y}$? Call this β .
What do you think are the values of $P\{\mathbf{Y} \leq \alpha - 1\}$ and $P\{\mathbf{Y} > 2\beta\}$?
- (b) What is the CDF $F_{\mathbf{Y}}(v)$ of the random variable $\mathbf{Y}$?
Be sure to specify the value of $F_{\mathbf{Y}}(v)$ for all v , $-\infty < v < \infty$.
- (c) Show that the CDF $F_{\mathbf{Y}}(v)$ that you found in part (b) is a nondecreasing continuous function. Is $F_{\mathbf{Y}}(v)$ differentiable at α ? at β ?
- (d) From the definition of the CDF $F_{\mathbf{Y}}(v)$, we know that $P\{\mathbf{Y} \leq \alpha - 1\} = F_{\mathbf{Y}}(\alpha - 1)$ and $P\{\mathbf{Y} > 2\beta\} = 1 - F_{\mathbf{Y}}(2\beta)$. Does substituting $v = \alpha - 1$ and $v = 2\beta$ in the CDF $F_{\mathbf{Y}}(v)$ that you found in part (b) give the same values for $P\{\mathbf{Y} \leq \alpha - 1\}$ and $P\{\mathbf{Y} > 2\beta\}$ that you stated in part (a)? If the values are different, which ones are the correct values? Explain your choices of correct answer (and why the other possible answers are wrong) in detail.
6. The radius of a sphere is a random variable $\mathbf{R}$ with pdf $f_{\mathbf{R}}(\rho) = \begin{cases} 3\rho^2, & 0 < \rho < 1, \\ 0, & \text{elsewhere.} \end{cases}$
- (a) Find the CDF $F_{\mathbf{V}}(\alpha)$ and pdf $f_{\mathbf{V}}(\alpha)$ of $\mathbf{V}$, the volume of the sphere.
- (b) If the sphere is made of metal and carries an electrical charge of Q coulombs, what is the CDF $F_{\mathbf{S}}(x)$ and the pdf $f_{\mathbf{S}}(x)$ of the surface charge density $\mathbf{S}$ on the sphere?