

ECE 313

FINAL EXAMINATION

Friday December 19, 1997, 8:00 a.m. - 11:00 a.m., 151 Everitt Laboratory

1. Check the appropriate box for each of the following statements. No justification is required, but, in order to discourage guessing, scoring will be as follows: You get +3 points for the right answer; 0 for no answer, and -2 points for a wrong answer.

(a) Let A and B denote events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.

TRUE FALSE

☐	☐	$P(A \cup B) = P(A^c \cap B^c) - P(A^c) - P(B^c) + 1$
☐	☐	$P(A \cap B) \geq P(A) + P(B) - 1$
☐	☐	$P(A \cap B) > P(B A)$
☐	☐	$P(A B)P(A) = P(B A)P(B)$
☐	☐	$P(A B) > P(A)$
☐	☐	$P(A B) = 1 - P(B A)$
☐	☐	$P(B A) + P(B A^c) = 1$
☐	☐	$P(A B) + P(A^c B) = 1$
☐	☐	If $P(A B) = P(B A)$, then A and B are independent events
☐	☐	If $P(A B) = P(B A)$, then $P(A) = P(B)$
☐	☐	If $P(A) = P(B)$, then $P(A B) = P(B A)$
☐	☐	$P(A B)P(B) + P(A B^c)P(B^c) = P(A)$
☐	☐	$P(A B)P(B) + P(A^c B)P(B) = P(B)$

(b) Which of the following statements are true for **all** random variables **X** and **Y** with identical finite variance σ^2 ?

TRUE FALSE

☐	☐	$E[X^2] = E[Y^2]$
☐	☐	$\text{var}(X + Y) = 2\sigma^2$
☐	☐	$\text{var}(X - Y) = 0$
☐	☐	$\text{var}(2X + 3Y) = \text{var}(3X + 2Y)$
☐	☐	$\text{cov}(X, Y) \leq \sigma^2$
☐	☐	X + Y and X - Y are uncorrelated random variables
☐	☐	X + Y and X - Y are independent random variables

2. Fred and Wilma take turns tossing a biased coin that turns up heads with probability p . If Fred tosses a head, he wins the game. If Wilma tosses a tail, she wins the game. The coin tossings continue until one of the two players wins. As usual, Fred tosses first.
- (a) Find the probability that Fred wins the game. (Hint: the answer is not p)
- (b) Now, find the probability that Wilma wins the game from first principles, that is, do not just write down $P\{\text{Wilma wins}\} = 1 - P\{\text{Fred wins}\} = 1 - \{\text{answer from part (a)}\}$
- (c) So, do your answers to parts (a) and (b) add up to 1? ☐ Yes ☐ No
- (d) For what value(s) of p is this a fair game (i.e. a game that both Fred and Wilma have equal probability $1/2$ of winning)?
- (e) Now consider once again the general game (with arbitrary value of p), and let $\mathbf{X}$ denote the total number of coin tosses in the game. Thus, $\mathbf{X}$ is a discrete random variable taking on values $1, 2, 3, \dots$ and in parts (a) and (b), you found $P\{\mathbf{X} \text{ is odd}\}$ and $P\{\mathbf{X} \text{ is even}\}$ respectively. Find $E[\mathbf{X}]$ as a function of p .

- 3.(a) A continuous random variable $\mathbf{X}$ has probability density function (pdf) given by

$$f_{\mathbf{X}}(u) = \begin{cases} \cos u, & -\pi/2 \leq u \leq 0, \\ 0, & \text{otherwise,} \end{cases}$$

Find its cumulative probability distribution function (CDF) $F_{\mathbf{X}}(u)$. To receive full credit, you must specify the value of $F_{\mathbf{X}}(u)$ for all u , $-\infty < u < \infty$.

- 3.(b) A continuous random variable $\mathbf{Y}$ is uniformly distributed on the interval $[a, b]$. It is known that $\mathbf{Y}$ has mean 2 and *standard deviation* $\sqrt{3}$. Find $P\{\mathbf{Y} \leq 1\}$.

4. Under hypothesis H_0 , a continuous random variable $\mathbf{X}$ has pdf

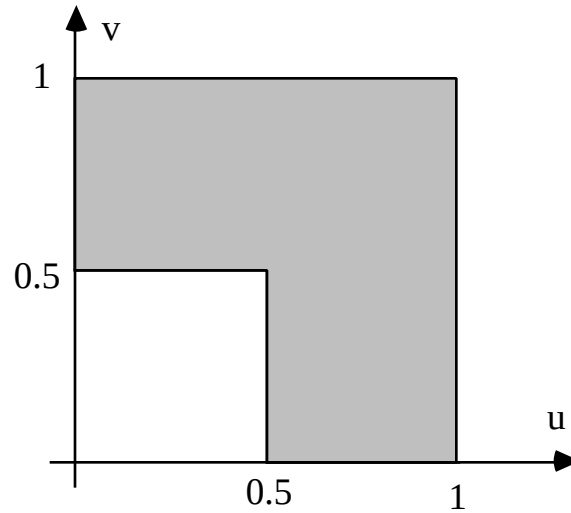
$$f_0(u) = (0.5) \exp(-|u + 5|), \quad -\infty < u < \infty$$

while under hypothesis H_1 , $\mathbf{X}$ has pdf

$$f_1(u) = (0.5) \exp(-|u - 5|), \quad -\infty < u < \infty$$

- (a) What is the maximum-likelihood decision rule, and what are the probabilities of false alarm (decision rule chooses H_1 when H_0 is true) and missed detection (decision rule chooses H_0 when H_1 is true) ?
- (b) Suppose that $P(H_0) = 1/3$ and $P(H_1) = 2/3$. Find the decision rule that minimizes the average error probability = $P(\text{decision rule chooses wrong hypothesis})$ and the corresponding minimum average error probability.

5. The random point (X, Y) is uniformly distributed on the shaded region shown below.



- (a) Find the marginal pdf $f_X(u)$ of the random variable X . In order to obtain full credit, you must specify the value of $f_X(u)$ for all real numbers u .
 - (b) Find $P\{X < Y < 2X\}$.
 - (c) Find the pdf of the random variable $Z = Y/X$. In order to obtain full credit, you must specify the value of $f_Z(\alpha)$ for all real numbers α , $-\infty < \alpha < \infty$.
6. The jointly Gaussian random variables X and Y have means 0 and 1 respectively, variances 4 and 16 respectively, and correlation coefficient $1/16$.
- (a) Find the probability density function of $Z = 5X + Y$. In order to obtain full credit, you must specify the value of $f_Z(w)$ for all real numbers w .
 - (b) Find $P\{X > Y\}$. A numerical value is desired.