University of Illinois

Solutions to Second Hour Exam

ECE 313
Spring 1999

Some numerical answers depend on which version of the exam you were solving.

1.(a) $P\{\mathbf{X} > \pi/4\} = \int_{\pi/4}^{\pi/2} f_{\mathbf{X}}(u)du = \frac{1}{2}\int_{\pi/4}^{\pi/2}\cos(u)du$ or $\frac{1}{2}\int_{\pi/4}^{\pi/2}\sin(u)du$ depending on the version of the exam. The numerical answers are $\frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$ and $\frac{1}{2\sqrt{2}}$.

(b) $E[\mathbf{Y}] = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos(u)|\sin(u)|du = \int_{0}^{\pi/2}\cos(u)\sin(u)du = \frac{1}{2}[\sin(u)]^2\Big|_{0}^{\pi/2} = \frac{1}{2}.$

2.(a) Let P(L) = p and P(D) = q = 1–p. Now, if the first box has a picture of Darth, then $\mathbf{X} = 1$ because Mrs Kirk can stop buying cereal as soon as the first picture of Luke is obtained. $\mathbf{X}$ also equals 1 if the first box had a picture of Luke and the second a picture of Darth. Thus, $\mathbf{X} = 1$ if and only if the first two boxes bought are LD, DL or DD. This has probability $1 - P(LL) = 1 - p^2$. On the other hand, for $n > 1$, $P\{\mathbf{X} = n\} = P(LLL\ldots LD) = p^n q$. Jimmy can check the result by verifying that our pmf sums to 1:

$$\sum_{n>1} p^n q = p^2q + p^3q + p^4q + \ldots = p^2q[1 + p + p^2 + \ldots] = p^2q/(1-p) = p^2 = 1 - P\{\mathbf{X} = 1\}$$

(b) $E[\mathbf{X}] = 1(1 - p^2) + 2p^2q + 3p^3q + 4p^4q + \ldots = (1 + p)(1 - p) + 2p^2q + 3p^3q + 4p^4q + \ldots$
$= q + pq + 2p^2q + 3p^3q + 4p^4q + \ldots = q + pq[1 + 2p + 3p^2 + \ldots] = q + \frac{pq}{(1-p)^2} = q + \frac{p}{q}.$
We can obtain the same result by the following "simpler" argument: The chance of getting a D in the first box is q in which case Jimmy gets only one picture of Luke. Else, with probability p, Jimmy gets a picture of Luke right away. He now waits for a picture of Darth, and eats (on average) an *additional* 1/q boxes, thereby getting (on average) an *additional* 1/q–1 pictures of Luke. Thus, the average number of pictures of Luke that he accumulates is $1\times q + (1/q)\times p = q + p/q$. Numerical answers are $1\frac{1}{6}$ and $2\frac{1}{3}$

3. $1 = \int_{-\infty}^{\infty} f_{\mathbf{X}}(u)du = \int_{0}^{1} a + bu\,du = au + bu^2/2\Big|_{0}^{1} = a + b/2.$

Also, $\mu = E[\mathbf{X}] = \int_{-\infty}^{\infty} uf_{\mathbf{X}}(u)du = \int_{0}^{1} au + bu^2\,du = au^2/2 + bu^3/3\Big|_{0}^{1} = a/2 + b/3$. Solve for a and b to get $a = 4 - 6\mu$, $b = 12\mu - 6$. The pdf is thus either 2u or 2(1–u) on [0, 1].
$P\{\mathbf{X} < 1/2\} = \int_{0}^{1/2} a + bu\,du = \frac{4a + b}{8} = \frac{5}{4} - \frac{3\mu}{2}$; the numerical answers are $\frac{3}{4}$ and $\frac{1}{4}$.

4.(a) $P\{\{|\mathbf{X} - 4| > 3\} = P\{\mathbf{X} > 7\} + P\{\mathbf{X} < 1\} = 1 - \Phi\left(\frac{7-2}{5}\right) + \Phi\left(\frac{1-2}{5}\right) = 1 - \Phi(1) + \Phi(-0.2)$
$= 1 - \Phi(1) + 1 - \Phi(0.2) = 2 - 0.8413 - 0.5793 = 0.5794.$

$P\{\mathbf{X} < 3 \mid \mathbf{X} > 2\} = P\{2 < \mathbf{X} < 3\}/P\{\mathbf{X} > 2\} = 2\left[\Phi\left(\frac{3-2}{5}\right) - \Phi\left(\frac{2-2}{5}\right)\right] = 2(\Phi(0.2) - 0.5)$
$= 1.1586 - 1 = 0.1586$

(b) For $v > 0$, $P\left\{\left(\frac{\mathbf{X} - 2}{5}\right)^2 < v\right\} = P\left\{-\sqrt{v} < \left(\frac{\mathbf{X} - 2}{5}\right) < \sqrt{v}\right\} = \Phi\left(\sqrt{v}\right) - \Phi\left(-\sqrt{v}\right).$

(c) $\frac{d}{dv}\Phi(v) = \exp(-u^2/2)/\sqrt{2\pi}$. Hence, the chain rule gives $f_{\mathbf{Y}}(v) = \exp(-v/2)/\sqrt{2\pi v}$ for $v > 0$ and 0 for $v < 0$. This is a gamma density with parameters (1/2, 1/2).