1. An experiment consists of tossing a fair coin repeatedly until a Head occurs for the tenth time.

(a) Suppose that the tenth Head occurred on the twentyfifth toss. What is the conditional probability that the first toss resulted in a Head?

(b) Let the random variable **X** denote the number of Tails that occur before the tenth Head occurs. For example, if the tenth Head occurred on the twentyfifth toss as was assumed in part (a), then **X** = 15 for this particular outcome of the experiment. More generally, **X** can take on any non-negative integer value 0, 1, 2, 3, … . What is P{**X** = k}?

1.(a) P{first toss is a Head | tenth Head on twentyfifth toss}

$$= \frac{P\{\text{first toss is a Head} \cap \text{tenth Head on twentyfifth toss}\}}{P\{\text{tenth Head on twentyfifth toss}\}} = \frac{P\{H;\ 8 \text{ Heads in 23 tosses}; H\}}{P\{\text{tenth Head on twentyfifth toss}\}}$$

$$= \frac{\left(\frac{1}{2}\right)\binom{23}{8}\left(\frac{1}{2}\right)^{8}\left(\frac{1}{2}\right)^{23-8}\left(\frac{1}{2}\right)}{\binom{24}{9}\left(\frac{1}{2}\right)^{25}} = \frac{\binom{23}{8}}{\binom{24}{9}} = \frac{23!9!15!}{24!8!15!} = \frac{9}{24} = \frac{3}{8}.$$

Alternatively, given that the tenth Head was on the 25th toss, we know for sure that nine Heads occurred on 24 tosses. Thus, the probability that the first toss resulted in a Head is 9/24 according to a formula derived in class.

(b) If **X** = k, then the tenth head necessarily must have occurred on the (k+10)-th toss. This has probability $\binom{k+9}{9}\left(\frac{1}{2}\right)^{k+10}$ of occurring, and hence P{**X** = k} $= \binom{k+9}{9}\left(\frac{1}{2}\right)^{k+10}$, k = 0, 1, 2, 3, … Notice, for example, that P{**X** = 0} = P{ten heads in a row} = 2^{-10} as one might expect.

2. RinkyDink Airways operates flights on eight-passenger planes between Champaign and Decatur. Exactly eight persons have reservations on a certain flight from Champaign to Decatur, and each passenger shows up with probability 3/4 for the flight. Assume also that each passenger decides independently of the others whether to show up for the flight.

(a) What is the probability that no passenger shows up for the flight?

Because the plane is needed in Decatur for the return flight, it must fly there even if there are no passengers on board. Furthermore, the pilot, refuses to fly with an odd number of passengers because the plane is not balanced on flights that have an odd number of passengers. Thus, if an odd number of passengers shows up for a flight, one of them (usually the last one to arrive at the gate) is denied boarding.

(b) What is the probability that the flight goes to Decatur with no passengers on board?

(c) What is the probability that one of the passengers who shows up is denied boarding?

2.(a) P{no passenger shows up for the flight} $= \left(\frac{1}{4}\right)^{8} = 2^{-16} = \frac{1}{65536}$

(b) P{flight goes with no passengers} = P{0 or 1 passenger shows up} $= \left(\frac{1}{4}\right)^{8} + \binom{8}{1}\left(\frac{1}{4}\right)^{7}\left(\frac{3}{4}\right) = \frac{25}{65536}$

(c) P{someone is denied boarding} = P{odd number of passengers show up} $= \frac{1}{2}\times\left[(q+p)^{8} - (q-p)^{8}\right]$ where q = 1/4, p = 3/4. Hence, P{someone is denied boarding} $= \frac{1}{2}\times(1-2^{-8}) = \frac{255}{512}$. If you didn't remember the formula from the homework, you can always compute P{odd number of passengers show up}

$$= \binom{8}{1}\left(\frac{1}{4}\right)^{7}\left(\frac{3}{4}\right) + \binom{8}{3}\left(\frac{1}{4}\right)^{5}\left(\frac{3}{4}\right)^{3} + \binom{8}{5}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{5} + \binom{8}{7}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{7}.$$

Since $\binom{8}{3} = \binom{8}{5} = 56$, you can readily compute the result as $\frac{8\times3 + 56\times3^{3} + 56\times3^{5} + 8\times3^{7}}{2^{16}} = \frac{8\times3\times(1 + 7\times9\times(1+3^{2}) + 3^{6})}{2^{16}}$

$= \frac{8\times3\times(1+630+729)}{2^{16}} = \frac{8\times3\times1360}{2^{16}} = \frac{3\times85}{2^{9}} = \frac{255}{512}$ without *really* needing a calculator!

3.(a) Three fair dice are rolled. The random variable **X** counts the number of dice showing the same face. For example, if the dice show 1, 4, and 3 respectively, **X** has value 0 while if the dice show 3, 4, and 3 respectively, then **X** has value 2. What is the probability mass function of **X**?

(b) **X** is a geometric random variable with parameter 1/4. What is the probability that the value of **X** is an even number?

3.(a) **X** can take on values 0, 2, and 3 only. $P\{\mathbf{X} = 3\} = P\{\text{all three dice show the same face}\} = 6\times\left(\frac{1}{6}\right)^3 = \frac{1}{36}$ since the probability that 2 (say) shows on all three dice is $\left(\frac{1}{6}\right)^3$, and there are 6 such possibilities. Similarly, $P\{\mathbf{X} = 2\} = 6\times\binom{3}{2}\times\left(\frac{1}{6}\right)^2\times\left(\frac{5}{6}\right) = \frac{15}{36} = \frac{5}{12}$ since there are 6 choices for which number shows on two of the dice and $\binom{3}{2}$ choices for which two of the three dice this number shows on. Finally, $P\{\mathbf{X} = 0\} = 6\times\frac{1}{6}\times\frac{5}{6}\times\frac{4}{6} = \frac{20}{36} = \frac{5}{9}$ because the number on the first die can be any of 6, and the other two dice must then be different. Reality check: 1 + 15 + 20 = 36, so there is no obvious error.

(b) $P\{\mathbf{X} \text{ is even}\} = P\{\mathbf{X} = 2, \text{ or } 4, \text{ or } 6, \text{ or } 8, \text{ or}\ldots\} = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right) + \ldots$

$= \left(\frac{3}{16}\right)\times\left[1 + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^4 + \ldots\right] = \left(\frac{3}{16}\right)\times\frac{1}{1 - 9/16} = \frac{3}{7}$. The formulaically unchallenged can even compute this as $P(\text{Betty wins}) = \frac{q}{q+1} = \frac{3/4}{1 + 3/4} = \frac{3}{7}$.

4. **X** is a continuous random variable with pdf $f_{\mathbf{X}}(u) = 0.5 \exp(-|u|)$, $-\infty < u < \infty$.

(a) What is the value of $P\{\mathbf{X} \le \ln 2\}$?

(b) Find the conditional probability that $P\{|\mathbf{X}| \le \ln 2\}$ given that $\{\mathbf{X} \le \ln 2\}$.

4.(a) From the figure shown below, we see that the pdf is symmetric about u = 0. Hence, we get that

$$P\{\mathbf{X} \le \ln 2\} = \frac{1}{2} + \int_0^{\ln 2} \frac{1}{2}\exp(-u)\, du = \frac{1}{2} + \left[-\frac{1}{2}\exp(-u)\right]_0^{\ln 2} = \frac{3}{4}.$$ Notice that $P\{0 \le \mathbf{X} \le \ln 2\} = \frac{1}{4}$

(b) $$P\{|\mathbf{X}| \le \ln 2 \mid \mathbf{X} \le \ln 2\} = \frac{P\{\{|\mathbf{X}| \le \ln 2\} \cap \{\mathbf{X} \le \ln 2\}\}}{P\{\mathbf{X} \le \ln 2\}} = \frac{P\{|\mathbf{X}| \le \ln 2\}}{P\{\mathbf{X} \le \ln 2\}} = \frac{2P\{0 \le \mathbf{X} \le \ln 2\}}{3/4} = \frac{1/2}{3/4} = \frac{2}{3}.$$