ECE 313

FIRST HOUR EXAMINATION

Wednesday March 3, 1999
7:00 p.m. — 8:00 p.m.
269 Everitt Laboratory

Name ______________________________ University ID # ______________________
Print in BLOCK CAPITALS

Section ☐ C 10 a.m. MWF

☐ D 11 a.m. MWF Signature ___________________________

INSTRUCTIONS

This exam is closed book; however, you may consult both sides of your 8.5"×11" sheet of notes. Calculators, laptop computers, Palm Pilots etc. may not be used. Note that the problems are not weighted equally and pace yourself accordingly.

Write your answers in the spaces provided. It is *not necessary* to convert answers that are simple common fractions (such as 10/32) into decimal fractions (10/32 = 0.3125), but you should simplify the fractions by cancelling common factors from the numerator and denominator (thus, write 5/16 instead of 10/32).

Please show all your work. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page.

GRADING

1.	15 points	______
2.	20 points	______
3.	20 points	______
4.	25 points	______
5.	20 points	______
TOTAL	**(100 points)**	______

This examination booklet consists of this cover page and four additional numbered pages

1. This problem has two unrelated parts. (15 points total)

(a) **(5 points)** Events A and B are **disjoint** events defined on a sample space. What is the probability that at least one of the two events A and B did not occur?

If the answer cannot be determined from the given information, check the square ☐ on the left side of the answer box and leave the right side of the answer box blank.

☐ Cannot be determined from the given data	P(at least one of A and B did not occur) =

(b) **(10 points)** Events A and B are events defined on a sample space. If the probability that at least one of the two events occurred is 0.6, and the probability that at least one of the events did not occur is 0.9, what is the probability that *exactly one* of the events A and B did not occur?

If the answer cannot be determined from the given information, check the square ☐ on the left side of the answer box and leave the right side of the answer box blank.

☐ Cannot be determined from the given data	P(*exactly one* of A and B did not occur) =

2 . **(20 points)** A, B, and C are events defined on a sample space. Suppose that A and B are *disjoint* events with $P(A) = 2P(B)$, and that $P(C|A) = 4/7$ and $P(C|B) = 2/7$. Find the value of $P(C|A \cup B)$.

If the answer cannot be determined from the given information, check the square □ on the left side of the answer box and leave the right side of the answer box blank.

□ Cannot be determined from the given data	$P(C\|A \cup B) =$

3 **(20 points)** Consider an experiment in which a fair coin is tossed until a head appears for the fifth time. Let **X** denote the number of tails that occur.

(a) **(8 points)** What are the values taken on by **X**?

X takes on values

(b) **(12 points)** What is the probability that **X** = 2?

$P\{\mathbf{X} = 2\} =$

4 . **(25 points)** Consider the following two part experiment.
A biased coin is tossed. If the outcome is a head (the event H), a ball is drawn at random from urn 1. If the outcome is a tail (the event T), a ball is drawn at random from urn 2. Urn 1 contains 3 red balls and 3 black balls, while urn 2 contains 4 red balls and 2 black balls. Let R denote the event that the ball drawn is red.

(a) **(10 points)** Suppose that $P(R) = 7/12$. Find the value of $P(H)$.

$P(H) =$

(b) **(15 points)** Now forget all about $P(R) = 7/12$ as stated in part (a) and suppose instead that $P(H) = 1/4$. Given that the event R^c occurred, find the probability that H occurred.

$P(H \mid R^c) =$

5 . **(20 points)** If hypothesis H_0 is true, **X** is a Poisson random variable with parameter λ_0, while if hypothesis H_1 is true, **X** is a Poisson random variable with parameter λ_1 where $\lambda_1 < \lambda_0$. The experiment is performed and it is observed that **X** took on value k.

(a) **(8 points)** What is the value of the likelihood ratio Λ when **X** has value k?

$\Lambda =$

(b) **(12 points)** The maximum-likelihood decision rule, which compares the likelihood ratio to 1, can be expressed as a threshold test on k. Find the threshold against which k is tested, and specify whether the decision is in favor of H_1 or H_0 when k exceeds this threshold. Thus, be sure to write 0 or 1 in the gray-shaded squares in the answer box.

$$k \underset{H_{\square}}{\overset{H_{\square}}{\gtrless}}$$

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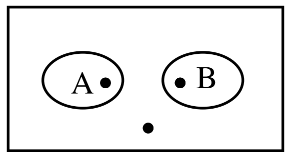
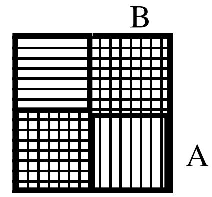
Numerical answers depend on which version of the exam you were solving

1.(a) It is easy to see from the left-hand diagram below that no matter what the outcome is, at least one of A and B did not occur. Hence, P(at least one of A and B did not occur) = 1. More formally,
P(at least one of A and B did not occur) = $P(A^c \cup B^c) = P((A \cap B)^c) = P(\Omega) = 1$ since $A \cap B = \varnothing$ regardless of the values of P(A) and P(B).





(b) In the right-hand diagram above, $A \cup B$, the event that at least one of A and B occurred, has vertical stripes while $A^c \cup B^c$, the event that at least one of A and B did not occur, has horizontal stripes. The union of these is obviously Ω. The forever plaid region is $A \oplus B = (A \cap B^c) \cup (A^c \cap B)$, the event that *exactly one* of A and B did not occur. Thus, we have that
$1 = P(\Omega) = P((A^c \cup B^c) \cup (A \cup B)) = P(A^c \cup B^c) + P(A \cup B) - P((A^c \cup B^c) \cap (A \cup B))$
$= P(A^c \cup B^c) + P(A \cup B) - P((A \cap B^c) \cup (A^c \cap B) = P(A^c \cup B^c) + P(A \cup B) - P(A \oplus B)$.
Hence, $P(A \oplus B) = P(A^c \cup B^c) + P(A \cup B) - 1$.

2. $P(C|A \cup B) = \dfrac{P(C \cap (A \cup B))}{P(A \cup B)} = \dfrac{P((C \cap A) \cup (C \cap B))}{P(A \cup B)} = \dfrac{P(C \cap A) + P(C \cap B)}{P(A) + P(B)}$ since A and B are disjoint. Now, $P(C \cap A) = P(C|A)P(A)$, $P(C \cap B) = P(C|B)P(B)$, and $P(A) = 2P(B)$ giving that

$$P(C|A \cup B) = \frac{P(C|A)P(A) + P(C|B)P(B)}{P(A) + P(B)} = \frac{P(C|A)2P(B) + P(C|B)P(B)}{2P(B) + P(B)} = \frac{2P(C|A) + P(C|B)}{3}$$

3.(a) Clearly, we could get a run of heads and the experiment could end without observing a tail at all, or we could get a run of heads interrupted by a single tail, etc. Hence, **X** can take on values 0, 1, 2, ...

(b) Let the probability of a head be p and the probability of a tail q. Then, P{**X** = k} = P{n+k tosses in which k tails and n-1 heads were observed on the first n+k-1 tosses and a head was observed on the (n+k)-th toss}
$= \binom{n+k-1}{k} \times p^{n-1} q^k \times p = \binom{n+k-1}{k} \times p^n q^k$ by independence of the tosses.

4.(a) $P(R) = P(R|H) \times P(H) + P(R|T) \times P(T) = P(R|H)P(H) + P(R|T) \times [1 - P(H)]$
$= P(R|T) + P(H) \times [P(R|H) - P(R|T)]$. Hence, $P(H) = \dfrac{P(R) - P(R|T)}{P(R|H) - P(R|T)}$.

(b) $P(H|R^c) = \dfrac{P(R^c|H)P(H)}{P(R^c)} = \dfrac{P(R^c|H)P(H)}{P(R^c|H)P(H) + P(R^c|T)P(T)}$.

5.(a) $\Lambda = \dfrac{\exp(-\lambda_1)(\lambda_1)^k/k!}{\exp(-\lambda_0)(\lambda_0)^k/k!} = \exp(-(\lambda_1 - \lambda_0))\left(\dfrac{\lambda_1}{\lambda_0}\right)^k$ where we note that $\lambda_1 - \lambda_0 < 0$ and $\lambda_1/\lambda_0 < 1$.

(b) $\Lambda = \exp(-(\lambda_1 - \lambda_0))\left(\dfrac{\lambda_1}{\lambda_0}\right)^k > 1$ if and only if $-(\lambda_1 - \lambda_0) + k \ln\left(\dfrac{\lambda_1}{\lambda_0}\right) > 0$. Note that the first term is positive while $\ln(\lambda_1/\lambda_0) < 0$. Hence, $\Lambda > 1$ if and only if $k < \dfrac{\lambda_1 - \lambda_0}{\ln(\lambda_1/\lambda_0)} = \dfrac{\lambda_0 - \lambda_1}{\ln(\lambda_0/\lambda_1)}$ where both terms on the right are positive. The maximum-likelihood decision rule can thus be expressed as $k \underset{H_1}{\overset{H_0}{\gtrless}} \dfrac{\lambda_0 - \lambda_1}{\ln(\lambda_0/\lambda_1)}$