

ECE 313: Exam I

Monday, October 12, 2026

7:00 p.m. — 8:30 p.m.

Name: (in BLOCK CAPITALS) _____

NetID: _____

Signature: _____

Section: ☐ A, MWF at 2 (Katselis) ☐ B, MWF at 10 (Zhao) ☐ C, MWF at 11 (Katselis)
☐ D, MWF at 1 (Katselis)

Instructions

This exam is closed book and closed notes except that one 8.5"×11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, microscopes, etc. are not allowed.

The exam consists of four problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. **Draw a small box around each of your final numerical answers.**

Grading

1. 20 points _____

2. 25 points _____

3. 25 points _____

4. 30 points _____

Total (100 points) _____

1. [20 points]

Suppose that a fair coin is independently tossed three times. Define the following events:

$$A = \{\text{The first and third tosses are heads}\},$$
$$B = \{\text{The second and third tosses are heads}\},$$
$$C = \{\text{All tosses yield the same outcome}\}.$$

(a) (10 pts) Are A , B , and C pairwise independent? Justify your answer.

(b) (10 pts) Are A , B , and C mutually independent? Justify your answer.

2. [25 points]

Suppose the random variable X has the pmf

$$p_X(k) = \begin{cases} c(4-k), & k = 1, 2, 3, \\ d, & k = m, \\ 0, & \text{otherwise,} \end{cases}$$

where $m \notin \{1, 2, 3\}$ is an integer.

- (a) (5 pts) Given that $p_X(m) = 2p_X(2)$, find the constants c and d so that p_X is a valid pmf.

- (b) (5 pts) Find m such that $\mathbb{E}[X] = -1$.

(c) (5 pts) Using the values from (a) and (b), find $\text{Var}(X)$.

(d) (10 pts) Let $Z = (X - 2)^2$. Find the pmf of Z and compute $\mathbb{E}[Z]$ in two ways: directly from the pmf of Z , and using only $\mathbb{E}[X]$ and $\text{Var}(X)$.

3. [25 points]

The number of requests X arriving at a web server in one second is a nonnegative integer-valued random variable with $\mathbb{E}[X] = 20$ and $\text{Var}(X) = 16$.

(a) (6 pts) Using only the mean, upper bound $P(X \geq 30)$.

(b) (6 pts) Using the mean and variance, upper bound $P(X \geq 30)$. Which bound is tighter?

(c) (6 pts) Find a lower bound on $P(12 < X < 28)$.

(d) (7 pts) Let $\bar{X}_n$ be the average (sample mean) of the request counts over n independent seconds, each distributed like X . How large must n be to guarantee, by Chebyshev's inequality, that $P(|\bar{X}_n - 20| \geq 1) \leq 0.01$?

4. [30 points]

A coin is flipped 4 times and the number of heads X is recorded. Under hypothesis H_0 the coin is fair, so $X \sim \text{Binomial}(4, \frac{1}{2})$. Under H_1 the coin is biased, so $X \sim \text{Binomial}(4, \frac{3}{4})$.

- (a) (10 pts) Complete the likelihood matrix (the table of $P(X = k | H_i)$ for $k = 0, \dots, 4$), compute the likelihood ratio $\Lambda(k) = P(X = k | H_1)/P(X = k | H_0)$, and show that any likelihood ratio test has the form “decide H_1 if $X \geq k^*$.”

- (b) (10 pts) Find the ML decision rule and compute $p_{\text{false alarm}}$ and p_{miss} .

- (c) (10 pts) Suppose the priors are $\pi_0 = P(H_0) = 0.8$ and $\pi_1 = P(H_1) = 0.2$. Find the MAP rule and its average error probability p_e . Compare with p_e for the ML rule under the same priors.