University of Illinois **Fall 2026**

ECE 313: Conflict Exam I

Tuesday, October 13, 2026
8:00 a.m. — 9:30 a.m.

1. **[20 points]**
Suppose that a fair coin is independently tossed four times. Define the following events:

$$A = \{\text{The outcomes of tosses 1 and 3 are heads}\},$$
$$B = \{\text{The outcomes of tosses 2 and 3 are heads}\},$$
$$C = \{\text{The outcomes of tosses 1, 2, and 3 are the same}\}.$$

(a) (10 pts) Are A, B, and C pairwise independent? Justify your answer.
Solution: The sample space consists of all sequences of four tosses,

$$\Omega = \{HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,$$
$$THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT\}.$$

Since the coin is fair and the tosses are independent, all sixteen outcomes are equally likely, each with probability $\frac{1}{16}$.
The three events are

$$A = \{HHHH, HHHT, HTHH, HTHT\},$$
$$B = \{HHHH, HHHT, THHH, THHT\},$$
$$C = \{HHHH, HHHT, TTTH, TTTT\}.$$

Therefore,

$$P(A) = P(B) = P(C) = \frac{4}{16} = \frac{1}{4}.$$

The pairwise intersections are

$$A \cap B = \{HHHH, HHHT\},$$
$$A \cap C = \{HHHH, HHHT\},$$
$$B \cap C = \{HHHH, HHHT\}.$$

Hence,

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{2}{16} = \frac{1}{8}.$$

However,

$$P(A)P(B) = P(A)P(C) = P(B)P(C) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}.$$

Since

$$\frac{1}{8} \neq \frac{1}{16},$$

none of the three pairs is independent.
Therefore,

$$\boxed{A, B, \text{ and } C \text{ are not pairwise independent.}}$$

(b) (10 pts) Are A, B, and C mutually independent? Justify your answer.

Solution: For three events to be mutually independent, they must be pairwise independent and satisfy

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

From part (a), the three events are not pairwise independent. Therefore, they cannot be mutually independent.

We can also verify this directly.

Since

$$A \cap B \cap C = \{HHHH, HHHT\},$$

we have

$$P(A \cap B \cap C) = \frac{2}{16} = \frac{1}{8}.$$

On the other hand,

$$P(A)P(B)P(C) = \left(\frac{1}{4}\right)^3 = \frac{1}{64}.$$

Thus,

$$P(A \cap B \cap C) = \frac{1}{8} \neq \frac{1}{64} = P(A)P(B)P(C).$$

Therefore,

$\boxed{A, B, \text{ and } C \text{ are not mutually independent.}}$

2. **[20 points]**

(a) (10 pts) Let $X \sim \text{Geo}(1/2)$. Find $\text{Var}(3X+1) + \mathbb{E}[X^2 + 2X]$. **Solution:** Since $X \sim \text{Geo}(1/2)$, we have

$$\mathbb{E}[X] = \frac{1}{p} = 2, \qquad \text{Var}(X) = \frac{1-p}{p^2} = 2.$$

Using the variance property $\text{Var}(aX+b) = a^2\text{Var}(X)$,

$$\text{Var}(3X+1) = 9\text{Var}(X) = 9(2) = 18.$$

Next, recall that

$$\mathbb{E}[X^2] = \text{Var}(X) + (\mathbb{E}[X])^2 = 2 + 2^2 = 6.$$

By linearity of expectation,

$$\mathbb{E}[X^2 + 2X] = \mathbb{E}[X^2] + 2\mathbb{E}[X] = 6 + 2(2) = 10.$$

Therefore,

$$\boxed{\text{Var}(3X+1) + \mathbb{E}[X^2 + 2X] = 18 + 10 = 28.}$$

(b) (10 pts) Four fair six-sided dice are rolled independently. Let X be the total number of dice showing a 5. Find $P(X \geq 1 \mid X \leq 3)$. **Solution:** Each die shows a 5 with probability 1/6, independently of the other dice. Therefore,

$$X \sim \text{Binomial}\left(4, \frac{1}{6}\right).$$

By the definition of conditional probability,

$$P(X \geq 1 \mid X \leq 3) = \frac{P(1 \leq X \leq 3)}{P(X \leq 3)}.$$

Since X takes values in $\{0, 1, 2, 3, 4\}$,

$$P(1 \leq X \leq 3) = 1 - P(X = 0) - P(X = 4),$$

and

$$P(X \leq 3) = 1 - P(X = 4).$$

Using the binomial pmf,

$$P(X = 0) = \binom{4}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^4 = \frac{625}{1296},$$

$$P(X = 4) = \binom{4}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^0 = \frac{1}{1296}.$$

Thus,

$$\begin{aligned} P(X \geq 1 \mid X \leq 3) &= \frac{1 - \frac{625}{1296} - \frac{1}{1296}}{1 - \frac{1}{1296}} \\ &= \frac{\frac{670}{1296}}{\frac{1295}{1296}} \\ &= \frac{670}{1295} \\ &= \boxed{\frac{134}{259} \approx 0.5174}. \end{aligned}$$

3. **[30 points]**

Let X and Y be two nonnegative random variables such that

$$\mathbb{E}[X] = 2, \qquad \mathbb{E}[Y] = 4, \qquad \mathrm{Var}(X) = 1, \qquad \mathrm{Var}(Y) = 2.$$

(a) (8 pts) Upper bound $P(X + Y \geq 10)$ using Markov's inequality.
Solution: Since X and Y are nonnegative, $X + Y$ is also nonnegative. By linearity of expectation,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 2 + 4 = 6.$$

By Markov's inequality with $a = 10$,

$$P(X + Y \geq 10) \leq \frac{\mathbb{E}[X + Y]}{10} = \frac{6}{10}.$$

Therefore,

$$\boxed{P(X + Y \geq 10) \leq \frac{3}{5}.}$$

(b) (8 pts) Provide an upper bound for $P(X^2 + Y^2 \geq 50)$.
Solution: The random variable $X^2 + Y^2$ is nonnegative, so Markov's inequality applies. First,

$$\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}[X])^2 = 1 + 4 = 5,$$
$$\mathbb{E}[Y^2] = \mathrm{Var}(Y) + (\mathbb{E}[Y])^2 = 2 + 16 = 18.$$

By linearity of expectation,

$$\mathbb{E}[X^2 + Y^2] = 5 + 18 = 23.$$

By Markov's inequality with $a = 50$,

$$\boxed{P(X^2 + Y^2 \geq 50) \leq \frac{\mathbb{E}[X^2 + Y^2]}{50} = \frac{23}{50}.}$$

(c) (8 pts) Upper bound $P(X \geq 4)$ using Chebyshev's inequality.
Solution: If $X \geq 4$, then $X - 2 \geq 2$, so $|X - 2| \geq 2$. Hence

$$P(X \geq 4) \leq P(|X - 2| \geq 2).$$

By Chebyshev's inequality,

$$P(|X - 2| \geq 2) \leq \frac{\mathrm{Var}(X)}{2^2} = \frac{1}{4}.$$

Therefore,

$$\boxed{P(X \geq 4) \leq \frac{1}{4}.}$$

(d) (6 pts) **True or False**: $P(Y < 6) \geq \frac{1}{3}$. Justify your answer.
Solution: **True.** Since Y is nonnegative, Markov's inequality with $a = 6$ gives

$$P(Y \geq 6) \leq \frac{\mathbb{E}[Y]}{6} = \frac{4}{6} = \frac{2}{3}.$$

Therefore,

$$P(Y < 6) = 1 - P(Y \geq 6) \geq 1 - \frac{2}{3} = \frac{1}{3}.$$

Alternatively (a stronger bound via Chebyshev): if $Y \geq 6$, then $|Y - 4| \geq 2$, so

$$P(Y \geq 6) \leq P(|Y - 4| \geq 2) \leq \frac{\mathrm{Var}(Y)}{2^2} = \frac{2}{4} = \frac{1}{2},$$

which gives $P(Y < 6) \geq \frac{1}{2} \geq \frac{1}{3}$.
Either way,

$$\boxed{\text{True: } P(Y < 6) \geq \frac{1}{3}.}$$

4. **[30 points]**

Consider a binary hypothesis testing problem with the following two hypotheses:

$$H_0 : X \sim \text{Geo}(1/2), \qquad H_1 : X \sim \text{Pois}(1).$$

(a) (7 pts) Find the maximum likelihood (ML) decision rule. **Solution:** Under the two hypotheses, the probability mass functions are

$$p_0(k) = \begin{cases} 2^{-k}, & k = 1, 2, \ldots, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p_1(k) = \begin{cases} \dfrac{e^{-1}}{k!}, & k = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

The ML decision rule chooses the hypothesis with the larger likelihood:

$$\widehat{H}_{\text{ML}}(k) = \begin{cases} H_1, & p_1(k) > p_0(k), \\ H_0, & p_1(k) < p_0(k). \end{cases}$$

For $k = 0$, we have $p_0(0) = 0$ and $p_1(0) = e^{-1} > 0$. Therefore, the ML rule chooses H_1.
For $k \geq 1$, the likelihood ratio is

$$\Lambda(k) = \frac{p_1(k)}{p_0(k)} = \frac{e^{-1}/k!}{2^{-k}} = \frac{2^k}{e\,k!}.$$

The ML rule chooses H_1 when

$$\Lambda(k) > 1 \quad \Longleftrightarrow \quad \frac{2^k}{k!} > e.$$

Evaluating the likelihood ratios,

$$\Lambda(1) = \frac{2}{e} < 1, \qquad \Lambda(2) = \frac{2}{e} < 1,$$

$$\Lambda(3) = \frac{4}{3e} < 1.$$

Moreover,

$$\frac{\Lambda(k+1)}{\Lambda(k)} = \frac{2}{k+1} < 1, \qquad k \geq 2.$$

Thus, $\Lambda(k) < 1$ for all $k \geq 1$.
Therefore, the ML decision rule is

$$\boxed{\widehat{H}_{\text{ML}}(k) = \begin{cases} H_1, & k = 0, \\ H_0, & k \geq 1. \end{cases}}$$

(b) (6 pts) Find the false alarm probability $p_{\text{false alarm}}$ and the miss probability p_{miss} for the ML decision rule. **Solution:** A false alarm occurs when the decision is H_1 while H_0 is true. Since the ML rule chooses H_1 only when $X = 0$,

$$p_{\text{false alarm}} = P(\widehat{H} = H_1 \mid H_0) = P(X = 0 \mid H_0).$$

Under H_0, $X \sim \text{Geo}(1/2)$, whose support is $\{1, 2, \ldots\}$. Hence,

$$\boxed{p_{\text{false alarm}} = 0.}$$

A miss occurs when the decision is H_0 while H_1 is true. The ML rule chooses H_0 when $X \geq 1$. Therefore,

$$\begin{aligned} p_{\text{miss}} &= P(\widehat{H} = H_0 \mid H_1) \\ &= P(X \geq 1 \mid H_1) \\ &= 1 - P(X = 0 \mid H_1) \\ &= 1 - e^{-1}. \end{aligned}$$

Thus,

$$\boxed{p_{\text{miss}} = 1 - \frac{1}{e} \approx 0.6321.}$$

(c) (7 pts) Find the maximum a posteriori (MAP) decision rule, assuming $\pi_0 = 4\pi_1$. **Solution:** The MAP rule chooses the hypothesis with the larger posterior probability. Equivalently, it chooses H_1 when

$$\pi_1 p_1(k) > \pi_0 p_0(k).$$

Since $\pi_0 = 4\pi_1$, this is equivalent to

$$\frac{p_1(k)}{p_0(k)} > 4,$$

whenever $p_0(k) > 0$.
For $k = 0$, $p_0(0) = 0$ and $p_1(0) > 0$, so the MAP rule chooses H_1.
For $k \geq 1$, the likelihood ratio satisfies

$$\Lambda(k) = \frac{2^k}{e\,k!} < 1 < 4.$$

Therefore, the MAP rule chooses H_0 for all $k \geq 1$.
The MAP decision rule is

$$\boxed{\widehat{H}_{\text{MAP}}(k) = \begin{cases} H_1, & k = 0, \\ H_0, & k \geq 1. \end{cases}}$$

Thus, the MAP and ML decision rules coincide for the given hypotheses and prior probabilities.

(d) (10 pts) Find the average probability of error, p_e, for both the ML and MAP decision rules using the prior probabilities given in part (c). **Solution:** The average probability of error is

$$p_e = \pi_0 p_{\text{false alarm}} + \pi_1 p_{\text{miss}}.$$

Since $\pi_0 = 4\pi_1$ and $\pi_0 + \pi_1 = 1$, we obtain

$$\pi_0 = \frac{4}{5}, \qquad \pi_1 = \frac{1}{5}.$$

From parts (a) and (b), the ML rule has

$$p_{\text{false alarm}} = 0, \qquad p_{\text{miss}} = 1 - e^{-1}.$$

Therefore, the average probability of error for the ML rule is

$$\begin{aligned} p_e^{\text{ML}} &= \pi_0 p_{\text{false alarm}} + \pi_1 p_{\text{miss}} \\ &= \frac{4}{5}(0) + \frac{1}{5}(1 - e^{-1}) \\ &= \boxed{\frac{1 - e^{-1}}{5} \approx 0.1264}. \end{aligned}$$

From part (c), the MAP rule coincides with the ML rule: both choose H_1 if $X = 0$ and H_0 otherwise.
Hence, the MAP rule has the same false alarm and miss probabilities, giving

$$\boxed{p_e^{\text{MAP}} = p_e^{\text{ML}} = \frac{1 - e^{-1}}{5} \approx 0.1264.}$$