ECE 313: Problem Set 9

Due: Friday, Nov. 20 at 06:59:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.1-4.8

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Sum of two independent geometric random variables]**
   Let X be a geometric random variable with parameter p, and Y be a geometric random variable with parameter q, with X and Y being independent. Let $S = X + Y$.

   (a) Suppose $0 < q < p < 1$. Find the pmf of S using convolution. Simplify your expression as much as possible (in particular, don't leave the answer as a summation).

   (b) Now consider the special case of $p = q$. Find the pmf of S using convolution.

   (c) Verify your answer to part (b) using the negative binomial distribution.

2. **[Sum of two independent continuous-type random variables]**
   Suppose X and Y have the joint pdf

$$f_{X,Y}(u,v) = \begin{cases} 2v & \text{if } 0 \le u \le 1,\ 0 \le v \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

   Let $S = X + Y$. Find the pdf of S, i.e., find $f_S(s)$. (*Hint:* You may want to consider the cases $s < 0$, $0 \le s < 1$, $1 \le s \le 2$, and $s > 2$ separately.)

3. **[Sum of two dependent continuous-type random variables]**
   The random variables X and Y with the following joint pdf are dependent:

$$f_{X,Y}(u,v) = \begin{cases} 6e^{-(u+2v)} & \text{if } 0 \le u < v, \\ 0 & \text{otherwise.} \end{cases}$$

   Find the pdf of the sum $S = X + Y$.

4. **[Discrete random variables]**
Two fair six-sided dice are rolled. One of the dice shows Z_1 pips, the other shows Z_2 pips. The random variables X and Y are defined as follows:

$$X = \min(Z_1, Z_2)$$
$$Y = \max(Z_1, Z_2)$$

   (a) Sketch the support, in the (u, v) plane, of the joint pmf $p_{X,Y}(u, v)$.
   (b) Find the joint pmf $p_{X,Y}(u, v)$.
   (c) Find the marginal pmf of Y.
   (d) Find $p_{X|Y}(u|v)$.
   (e) Find the joint CDF, $F_{X,Y}(c, d)$.
   (f) Find $E[Y - X]$.

5. **[Continuous random variables]**
Consider a pair of continuous-valued random variables, X and Y, whose pdf is given by the following pyramid, for some constant height A:

$$f_{X,Y}(u, v) = \begin{cases} Au & 0 \le u \le v \le 1 \\ Av & 0 \le v \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of the constant A?
   (b) Find the marginal pdf of Y.
   (c) Find $f_{X|Y}(u|v)$.
   (d) Find the CDF, $F_{X,Y}(c, d)$.
   (e) Find $E[X + Y]$.

6. **[Correlation of continuous-type random variables]**
Suppose X and Y are jointly distributed with the following joint pdf:

$$f_{X,Y}(u, v) = \begin{cases} \frac{3}{2}(1 - |u - 1| - |v|), & \text{if } -1 \le v \le 1 \text{ and } |v| \le u \le 2 - |v|, \\ 0, & \text{else.} \end{cases}$$

   (a) Sketch the support of $f_{X,Y}$, and describe the shape of $f_{X,Y}$ on the support.
   (b) Find $Cov(X, Y)$ and $\rho_{X,Y}$.
   (c) Are X and Y uncorrelated? Are X and Y independent?
   (d) Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$. Express $\rho_{W,Z}$ with respect to a, b, c, and d. (Hint: You may use the fact $Var(X) = Var(Y)$, which should be clear from your answer to part (a). You do not need to calculate $Var(X)$.)

7. **[Correlation of random variables with a symmetric joint pdf]**
Suppose X and Y have joint pdf $f_{X,Y}(u, v)$ which is symmetric with respect to line $u = u_0$ for some constant u_0. Show that X and Y are uncorrelated. (Hint: You may need (i) $Cov(X - u_0, Y) = Cov(X, Y)$ and (ii) $\int_{-\infty}^{\infty} ug(u)\, du = 0$ for any even function $g(u)$.)

8. **[The covariance of sums of correlated random variables]**
Suppose $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are random variables on a common probability space such that $Var(X_i) = Var(Y_i) = 4$ for all i, and

$$\rho(X_i, Y_j) = \begin{cases} \frac{3}{4} & \text{if } i = j, \\ -\frac{1}{2} & \text{if } i \neq j. \end{cases}$$

Let $W = \sum_{i=1}^{n} X_i$ and $Z = \sum_{i=1}^{n} Y_i$. Express $Cov(W, Z)$ as a function of n.