ECE 313: Problem Set 8: Problems and Solutions

Due: Friday, Nov. 6 at 06:59:00 p.m.

Reading: *ECE 313 Course Notes,* Sections 3.10, 4.1-4.2

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
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SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Binary hypothesis testing with exponential observations]**
   A quality-control engineer measures the lifetime X (in thousands of hours) of one component drawn from a production batch. Under H_0 (good batch), X is exponentially distributed with parameter $\lambda_0 = 1$. Under H_1 (defective batch), X is exponentially distributed with parameter $\lambda_1 = 3$.

   (a) Find the likelihood ratio $\Lambda(u)$ for $u \geq 0$. Show that the ML rule can be written as a threshold test on X, and find the threshold γ_{ML}. In which direction does the test go, and why?

   Solution: For $u \geq 0$, $f_1(u) = 3e^{-3u}$ and $f_0(u) = e^{-u}$, so

   $$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{3e^{-3u}}{e^{-u}} = 3e^{-2u}.$$

   Λ is decreasing in u: short lifetimes are evidence for the defective batch. The ML rule declares H_1 when $\Lambda(X) > 1$, i.e., $3e^{-2X} > 1$, i.e., $X < \frac{1}{2}\ln 3$:

   $$\text{declare } H_1 \text{ if } X < \gamma_{ML}, \qquad \text{declare } H_0 \text{ if } X > \gamma_{ML},$$

   where $\gamma_{ML} = \frac{\ln 3}{2} \approx 0.5493$.

   (b) Find $p_{\text{false alarm}}$ and p_{miss} for the ML rule.

   Solution:

   $$p_{\text{false alarm}} = P(X < \gamma_{ML} \mid H_0) = 1 - e^{-\gamma_{ML}} = 1 - 3^{-1/2} \approx 0.4226,$$
   $$p_{\text{miss}} = P(X > \gamma_{ML} \mid H_1) = e^{-3\gamma_{ML}} = 3^{-3/2} \approx 0.1925.$$

Unlike the Gaussian equal-variance case (Example 3.10.1), the two ML error probabilities are not equal.

(c) Suppose $\pi_1 = 0.4$ and $\pi_0 = 0.6$. Find the MAP rule and its error probabilities, and compute the average error probability p_e. Compare with p_e for the ML rule under the same prior.

Solution: The MAP threshold is $\tau = \pi_0/\pi_1 = 1.5$. Declare H_1 when $3e^{-2X} > 1.5$, i.e., $X < \gamma_{MAP} = \frac{1}{2}\ln 2 \approx 0.3466$. Then

$$p_{\text{false alarm}} = 1 - e^{-\gamma_{MAP}} = 1 - 2^{-1/2} \approx 0.2929,$$
$$p_{\text{miss}} = e^{-3\gamma_{MAP}} = 2^{-3/2} \approx 0.3536,$$
$$p_e = 0.6(0.2929) + 0.4(0.3536) \approx 0.3172.$$

For the ML rule with the same prior, $p_e = 0.6(0.4226) + 0.4(0.1925) \approx 0.3306$. The MAP rule has smaller average error probability, as it must, since the MAP rule minimizes p_e for the given prior. Because H_0 is more likely a priori, the MAP rule needs stronger evidence (a shorter lifetime) before declaring H_1.

(d) Now suppose $\pi_1 = 0.2$. Show that the MAP rule declares H_0 for every possible observation. Find $p_{\text{false alarm}}$, p_{miss} and p_e for this rule, and explain intuitively why this happens.

Solution: Now $\tau = \pi_0/\pi_1 = 4$. But $\Lambda(u) = 3e^{-2u} \le 3 < 4$ for all $u \ge 0$, so $\Lambda(X) > \tau$ never happens: the MAP rule always declares H_0. Then $p_{\text{false alarm}} = 0$, $p_{\text{miss}} = 1$ and $p_e = \pi_1 = 0.2$. A single lifetime measurement is never strong enough evidence (the likelihood ratio is at most 3) to overcome prior odds of 4 to 1 in favor of H_0. Any rule that sometimes declares H_1 has $p_e > 0.2$ under this prior.

(e) Suppose instead that the engineer requires $p_{\text{false alarm}} \le 0.05$. Among tests of the form "declare H_1 if $X < \gamma$", find the γ that minimizes p_{miss} subject to this constraint, and the resulting p_{miss}. Is this test an LRT?

Solution: $p_{\text{false alarm}} = 1 - e^{-\gamma}$ increases with γ while $p_{\text{miss}} = e^{-3\gamma}$ decreases, so choose the largest γ with $1 - e^{-\gamma} \le 0.05$:

$$\gamma = -\ln 0.95 \approx 0.0513, \qquad p_{\text{miss}} = e^{-3\gamma} = (0.95)^3 \approx 0.8574.$$

Since Λ is strictly decreasing, $\{X < \gamma\} = \{\Lambda(X) > 3e^{-2\gamma}\}$, so this is an LRT with threshold $\tau = 3(0.95)^2 \approx 2.71$. A strict false-alarm requirement forces a very large miss probability.

2. **[Binary hypothesis testing: noise versus interference]**
A receiver observes a single sample X. Under H_0 only thermal noise is present and X has the $\mathcal{N}(0,1)$ distribution. Under H_1 a jammer is active and X is uniformly distributed over $[-2, 2]$.

(a) Find the likelihood ratio $\Lambda(u)$ for all $u \in \mathbb{R}$. Be careful about the region $|u| > 2$.

Solution: $f_1(u) = \frac{1}{4}$ for $|u| \le 2$ and $f_1(u) = 0$ otherwise; $f_0(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$ for all u. Hence

$$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \begin{cases} \dfrac{\sqrt{2\pi}}{4}\, e^{u^2/2}, & |u| \le 2, \\ 0, & |u| > 2. \end{cases}$$

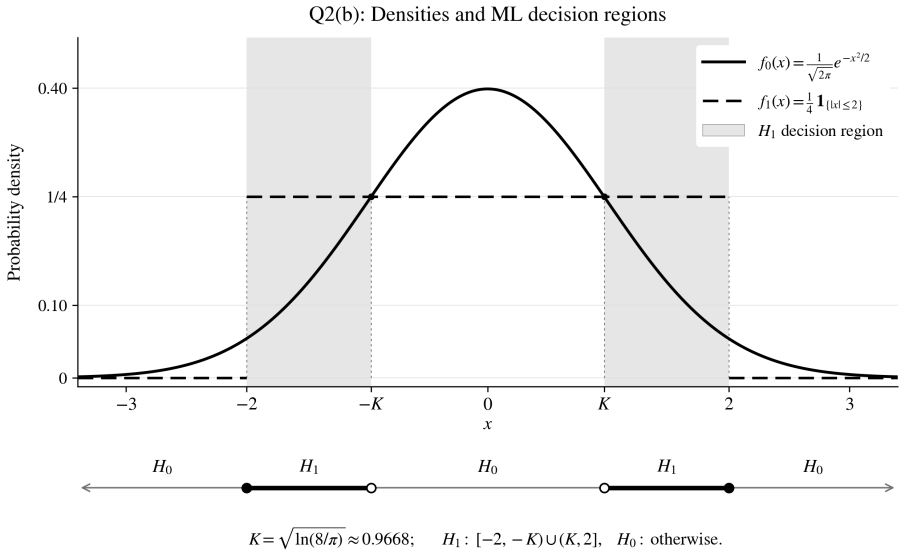
An observation with $|X| > 2$ is impossible under H_1, so it proves H_0 is true.

(b) Find the ML decision rule. Show that the set of observations for which H_1 is declared is a union of two intervals, and sketch f_0, f_1 and the decision regions.

Solution: For $|u| > 2$, $\Lambda(u) = 0 < 1$, so declare H_0. For $|u| \le 2$, $\Lambda(u) > 1$ if and only if $e^{u^2/2} > \frac{4}{\sqrt{2\pi}}$, i.e.,

$$u^2 > 2\ln\frac{4}{\sqrt{2\pi}} = \ln\frac{8}{\pi} \approx 0.9347,$$

i.e., $|u| > K$, where $K = \sqrt{\ln(8/\pi)} \approx 0.967$. So the ML rule is: declare H_1 if $K < |X| \le 2$, and declare H_0 otherwise. The H_1 region is $[-2, -K) \cup (K, 2]$. In the sketch, the flat pdf $f_1 = 1/4$ lies above the Gaussian bell exactly on these two intervals; near zero the bell is taller, and beyond ± 2 the uniform pdf is zero.



(c) Find $p_{\text{false alarm}}$ and p_{miss} for the ML rule.

Solution: Using symmetry of the $\mathcal{N}(0,1)$ pdf,

$$\begin{aligned} p_{\text{false alarm}} &= P(K < |X| \le 2 \mid H_0) = 2\big(Q(K) - Q(2)\big) \\ &\approx 2(0.1668 - 0.0228) \approx 0.288. \end{aligned}$$

Under H_1, $|X| > 2$ has probability zero, so

$$p_{\text{miss}} = P(|X| \le K \mid H_1) = \frac{2K}{4} = \frac{K}{2} \approx 0.483.$$

(d) Suppose $\pi_1 = 2/3$. Find the MAP rule, its error probabilities, and p_e. Compare with p_e for the ML rule under the same prior.

Solution: $\tau = \pi_0/\pi_1 = 1/2$. For $|u| \le 2$, $\Lambda(u) \ge \frac{\sqrt{2\pi}}{4} \approx 0.627 > \frac{1}{2}$, so H_1 is declared for every $|u| \le 2$; for $|u| > 2$, $\Lambda(u) = 0$ and H_0 is declared. The MAP rule is: declare H_1 if $|X| \le 2$. Then

$$\begin{aligned} p_{\text{false alarm}} &= P(|X| \le 2 \mid H_0) = 1 - 2Q(2) \approx 0.9545, \qquad p_{\text{miss}} = 0, \\ p_e &= \tfrac{1}{3}(0.9545) \approx 0.318. \end{aligned}$$

For the ML rule under this prior, $p_e = \frac{1}{3}(0.288) + \frac{2}{3}(0.483) \approx 0.418$, so the MAP rule does better, as expected.

(e) For what range of π_1 does the MAP rule declare H_0 for every observation?

Solution: The largest value of Λ is at $|u| = 2$: $\Lambda_{\max} = \frac{\sqrt{2\pi}}{4}e^2 \approx 4.630$. The MAP rule never declares H_1 when $\tau = \pi_0/\pi_1 \geq \Lambda_{\max}$, i.e., $\frac{1-\pi_1}{\pi_1} \geq 4.630$, i.e.,

$$\pi_1 \leq \frac{1}{1+4.630} \approx 0.178.$$

3. **[Joint CDFs]**
Part (a) concerns a separate function G. For parts (b)–(f), let X and Y take values in $[0,1]$ and have joint CDF satisfying

$$F_{X,Y}(u,v) = \frac{u^2v + uv^2}{2} \qquad \text{for } (u,v) \in [0,1]^2.$$

(a) Consider the function $G(u,v) = 1$ if $u+v \geq 1$ and $G(u,v) = 0$ otherwise. Show that G is nondecreasing in each variable, right-continuous in each variable, and has the correct limits, but is *not* a valid joint CDF.

Solution: Increasing u or v can only change $u+v < 1$ into $u+v \geq 1$, so G is nondecreasing in each variable; the set $\{u+v \geq 1\}$ is closed, so G is right-continuous. $G(u,v) = 0$ once $u < 1-v$, so $G \to 0$ as $u \to -\infty$ (and as $v \to -\infty$), and $G = 1$ for large u, v. However, for the rectangle $(0,1] \times (0,1]$,

$$G(1,1) - G(1,0) - G(0,1) + G(0,0) = 1 - 1 - 1 + 0 = -1 < 0.$$

By (4.1) this would be the probability that (X,Y) lies in the rectangle, which cannot be negative. So G is not a joint CDF.

(b) Express $F_{X,Y}(u,v)$ for all $(u,v) \in \mathbb{R}^2$. Then find the marginal CDFs F_X and F_Y.

Solution: Since $X, Y \in [0,1]$, $F_{X,Y}(u,v) = 0$ if $u < 0$ or $v < 0$, and for $u, v \geq 0$, $F_{X,Y}(u,v) = F_{X,Y}(\min(u,1), \min(v,1))$. In particular, $F_{X,Y}(u,v) = \frac{u^2+u}{2}$ for $0 \leq u \leq 1$, $v \geq 1$ and $F_{X,Y}(u,v) = 1$ for $u, v \geq 1$. By Proposition 4.1.1,

$$F_X(u) = F_{X,Y}(u,\infty) = \begin{cases} 0, & u < 0, \\ \frac{u^2+u}{2}, & 0 \leq u \leq 1, \\ 1, & u > 1, \end{cases}$$

and by symmetry of the formula in u and v, F_Y has the same form.

(c) Find $P\{X \leq \frac{1}{2},\ Y \leq \frac{1}{2}\}$ and $F_{X,Y}(2, 0.5)$.

Solution: $P\{X \leq \frac{1}{2}, Y \leq \frac{1}{2}\} = F_{X,Y}(\frac{1}{2}, \frac{1}{2}) = \frac{1/8+1/8}{2} = 0.125$. Also $F_{X,Y}(2, 0.5) = F_{X,Y}(1, 0.5) = \frac{0.5+0.25}{2} = 0.375$, which equals $F_Y(0.5)$.

(d) Find $P\{(X,Y) \in (\frac{1}{2}, 1] \times (\frac{1}{2}, 1]\}$ and $P\{X \leq \frac{1}{2},\ Y > \frac{1}{2}\}$.

Solution: By (4.1),

$$\begin{aligned} P\{(X,Y) \in (\tfrac{1}{2}, 1] \times (\tfrac{1}{2}, 1]\} &= F(1,1) - F(1, \tfrac{1}{2}) - F(\tfrac{1}{2}, 1) + F(\tfrac{1}{2}, \tfrac{1}{2}) \\ &= 1 - 0.375 - 0.375 + 0.125 = 0.375. \end{aligned}$$

Also, $P\{X \leq \frac{1}{2}, Y > \frac{1}{2}\} = F_X(\frac{1}{2}) - F(\frac{1}{2}, \frac{1}{2})$, which equals $0.375 - 0.125 = 0.25$. As a check, the four quadrant probabilities 0.125, 0.25, 0.25 and 0.375 sum to one.

(e) Are X and Y independent? Justify your answer.

Solution: If they were independent, $F_{X,Y}(u,v) = F_X(u)F_Y(v)$ for all u, v. But $F_{X,Y}(\frac{1}{2}, \frac{1}{2}) = 0.125$, while $F_X(\frac{1}{2})\,F_Y(\frac{1}{2}) = (0.375)^2 \approx 0.1406$. So X and Y are not independent.

(f) Let $M = \max(X, Y)$ and $L = \min(X, Y)$. Find $F_M(c)$ and $F_L(c)$ for $0 \le c \le 1$, and evaluate both at $c = \frac{1}{2}$.

Solution: $\{M \le c\} = \{X \le c, Y \le c\}$, so $F_M(c) = F_{X,Y}(c,c) = c^3$. For the minimum, use the complement and inclusion–exclusion:

$$\begin{aligned} F_L(c) &= 1 - P\{X > c, Y > c\} = F_X(c) + F_Y(c) - F_{X,Y}(c,c) \\ &= c^2 + c - c^3. \end{aligned}$$

At $c = \frac{1}{2}$: $F_M(\frac{1}{2}) = 0.125$ and $F_L(\frac{1}{2}) = 0.25 + 0.5 - 0.125 = 0.625$, which agrees with $1 - 0.375$ from part (d).

4. **[Joint pmfs]**

Let X and Y be integer-valued random variables with joint pmf

$$p_{X,Y}(i,j) = \begin{cases} c\,2^{-j}, & 1 \le i \le j, \quad i, j \text{ integers}, \\ 0, & \text{else.} \end{cases}$$

(a) Find the constant c.

Solution: For each $j \ge 1$ there are j values of i, so

$$\sum_{j=1}^{\infty} \sum_{i=1}^{j} c\,2^{-j} = c \sum_{j=1}^{\infty} j\,2^{-j} = c \cdot \frac{1/2}{(1-1/2)^2} = 2c.$$

(We used $\sum_{j \ge 1} j x^j = \frac{x}{(1-x)^2}$ for $|x| < 1$.) The pmf must sum to one, so $2c = 1$, $c = \frac{1}{2}$, and $p_{X,Y}(i,j) = 2^{-(j+1)}$ for $1 \le i \le j$.

(b) Find the marginal pmf p_Y and $E[Y]$.

Solution: Summing over i (the j terms with $1 \le i \le j$):

$$p_Y(j) = j\,2^{-(j+1)}, \qquad j = 1, 2, \ldots$$

Using $\sum_{j \ge 1} j^2 x^j = \frac{x(1+x)}{(1-x)^3}$, which equals 6 at $x = \frac{1}{2}$, $E[Y] = \frac{1}{2} \sum_j j^2 2^{-j} = 3$. (Equivalently, $P\{Y + 1 = n\} = (n-1)2^{-n}$ for $n \ge 2$, so $Y + 1$ has the negative binomial distribution with $r = 2$, $p = \frac{1}{2}$, and mean $r/p = 4$.)

(c) Find the marginal pmf p_X and identify it.

Solution: Summing over $j \ge i$:

$$p_X(i) = \sum_{j=i}^{\infty} 2^{-(j+1)} = 2^{-(i+1)} \cdot \frac{1}{1-1/2} = 2^{-i}, \qquad i = 1, 2, \ldots$$

So X is geometrically distributed with parameter $\frac{1}{2}$, and $E[X] = 2$.

(d) Find the conditional pmfs $p_{X|Y}(i \mid j)$ and $p_{Y|X}(j \mid i)$. Describe each in words.

Solution: For $j \ge 1$,

$$p_{X|Y}(i \mid j) = \frac{2^{-(j+1)}}{j\,2^{-(j+1)}} = \frac{1}{j}, \qquad i = 1, \ldots, j,$$

so given $Y = j$, X is uniformly distributed over $\{1, \ldots, j\}$. For $i \geq 1$,

$$p_{Y|X}(j \mid i) = \frac{2^{-(j+1)}}{2^{-i}} = 2^{-(j-i+1)}, \qquad j = i, i+1, \ldots,$$

so given $X = i$, $Y - i + 1$ is geometric with parameter $\frac{1}{2}$. One way to generate (X, Y): flip a fair coin until the first head (X flips), then keep flipping until the next head, and let Y be the total number of flips minus one.

(e) Find $P\{X = Y\}$ and $P\{Y \geq 2X\}$.

Solution:

$$P\{X = Y\} = \sum_{j=1}^{\infty} p_{X,Y}(j, j) = \sum_{j=1}^{\infty} 2^{-(j+1)} = \frac{1}{2}.$$

For the second, condition on X: $P(Y \geq 2i \mid X = i)$ equals $\sum_{j \geq 2i} 2^{-(j-i+1)} = 2^{-i}$, so by the law of total probability

$$P\{Y \geq 2X\} = \sum_{i=1}^{\infty} p_X(i)\, 2^{-i} = \sum_{i=1}^{\infty} 4^{-i} = \frac{1}{3}.$$

(f) Find the joint CDF value $F_{X,Y}(2, 3)$.

Solution: $F_{X,Y}(2,3) = P\{X \leq 2, Y \leq 3\}$ sums the pmf over the points $(1,1)$, $(1,2)$, $(2,2)$, $(1,3)$ and $(2,3)$:

$$F_{X,Y}(2,3) = \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{16} = \tfrac{5}{8} = 0.625.$$

(g) Are X and Y independent? Justify your answer.

Solution: No. For example, $p_{X,Y}(2,1) = 0$ because $i \leq j$ is violated, but $p_X(2)p_Y(1) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} \neq 0$. More generally, the support $\{1 \leq i \leq j\}$ is not a product set, which already rules out independence.