University of Illinois **Fall 2026**

ECE 313: Problem Set 8

Due: Friday, Nov. 6 at 06:59:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.10, 4.1-4.2

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Binary hypothesis testing with exponential observations]**
   A quality-control engineer measures the lifetime X (in thousands of hours) of one component drawn from a production batch. Under H_0 (good batch), X is exponentially distributed with parameter $\lambda_0 = 1$. Under H_1 (defective batch), X is exponentially distributed with parameter $\lambda_1 = 3$.

   (a) Find the likelihood ratio $\Lambda(u)$ for $u \geq 0$. Show that the ML rule can be written as a threshold test on X, and find the threshold γ_{ML}. In which direction does the test go, and why?

   (b) Find $p_{\text{false alarm}}$ and p_{miss} for the ML rule.

   (c) Suppose $\pi_1 = 0.4$ and $\pi_0 = 0.6$. Find the MAP rule and its error probabilities, and compute the average error probability p_e. Compare with p_e for the ML rule under the same prior.

   (d) Now suppose $\pi_1 = 0.2$. Show that the MAP rule declares H_0 for every possible observation. Find $p_{\text{false alarm}}$, p_{miss} and p_e for this rule, and explain intuitively why this happens.

   (e) Suppose instead that the engineer requires $p_{\text{false alarm}} \leq 0.05$. Among tests of the form "declare H_1 if $X < \gamma$", find the γ that minimizes p_{miss} subject to this constraint, and the resulting p_{miss}. Is this test an LRT?

2. **[Binary hypothesis testing: noise versus interference]**
   A receiver observes a single sample X. Under H_0 only thermal noise is present and X has the $\mathcal{N}(0,1)$ distribution. Under H_1 a jammer is active and X is uniformly distributed over $[-2, 2]$.

(a) Find the likelihood ratio $\Lambda(u)$ for all $u \in \mathbb{R}$. Be careful about the region $|u| > 2$.

(b) Find the ML decision rule. Show that the set of observations for which H_1 is declared is a union of two intervals, and sketch f_0, f_1 and the decision regions.

(c) Find $p_{\text{false alarm}}$ and p_{miss} for the ML rule.

(d) Suppose $\pi_1 = 2/3$. Find the MAP rule, its error probabilities, and p_e. Compare with p_e for the ML rule under the same prior.

(e) For what range of π_1 does the MAP rule declare H_0 for every observation?

3. **[Joint CDFs]**
Part (a) concerns a separate function G. For parts (b)–(f), let X and Y take values in $[0, 1]$ and have joint CDF satisfying

$$F_{X,Y}(u, v) = \frac{u^2 v + u v^2}{2} \qquad \text{for } (u, v) \in [0, 1]^2.$$

(a) Consider the function $G(u, v) = 1$ if $u + v \geq 1$ and $G(u, v) = 0$ otherwise. Show that G is nondecreasing in each variable, right-continuous in each variable, and has the correct limits, but is *not* a valid joint CDF.

(b) Express $F_{X,Y}(u, v)$ for all $(u, v) \in \mathbb{R}^2$. Then find the marginal CDFs F_X and F_Y.

(c) Find $P\{X \leq \frac{1}{2},\ Y \leq \frac{1}{2}\}$ and $F_{X,Y}(2, 0.5)$.

(d) Find $P\{(X, Y) \in (\frac{1}{2}, 1] \times (\frac{1}{2}, 1]\}$ and $P\{X \leq \frac{1}{2},\ Y > \frac{1}{2}\}$.

(e) Are X and Y independent? Justify your answer.

(f) Let $M = \max(X, Y)$ and $L = \min(X, Y)$. Find $F_M(c)$ and $F_L(c)$ for $0 \leq c \leq 1$, and evaluate both at $c = \frac{1}{2}$.

4. **[Joint pmfs]**
Let X and Y be integer-valued random variables with joint pmf

$$p_{X,Y}(i, j) = \begin{cases} c\, 2^{-j}, & 1 \leq i \leq j, \quad i, j \text{ integers}, \\ 0, & \text{else.} \end{cases}$$

(a) Find the constant c.

(b) Find the marginal pmf p_Y and $E[Y]$.

(c) Find the marginal pmf p_X and identify it.

(d) Find the conditional pmfs $p_{X|Y}(i \mid j)$ and $p_{Y|X}(j \mid i)$. Describe each in words.

(e) Find $P\{X = Y\}$ and $P\{Y \geq 2X\}$.

(f) Find the joint CDF value $F_{X,Y}(2, 3)$.

(g) Are X and Y independent? Justify your answer.