ECE 313: Problem Set 7: Problems and Solutions

Due: Friday, Oct. 30 at 06:59:00 p.m.

Reading: *ECE 313 Course Notes,* Sections 3.5 - 3.8

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
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SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Poisson process probabilities]**
   Consider a Poisson process with arrival rate $\lambda > 0$.

   (a) Find the probability there is exactly one arrival in each of the intervals $(0, 1]$, $(1, 2]$, and $(2, 3]$.

   Solution: The numbers of arrivals in the disjoint intervals are independent Poisson random variables with mean λ. Thus, the probability is

   $$\left(\lambda e^{-\lambda}\right)^3 = \lambda^3 e^{-3\lambda}.$$

   (b) Find the probability that there are two arrivals in the interval $(0, 2]$ and two arrivals in the interval $(1, 3]$. (*Hint:* The occurrence of the event in (a) implies the occurrence of the event in (b). What other ways can the event in (b) occur?)

   Solution: We wish to determine the probability of the event

   $$A = \{N(0, 2] = 2,\ N(1, 3] = 2\}.$$

   Note that A is the union of the mutually exclusive events B_{020}, B_{111}, and B_{202}, where

   $$B_{ijk} = \{N(0, 1] = i,\ N(1, 2] = j,\ N(2, 3] = k\}.$$

Since the events $\{N(0,1] = i\}$, $\{N(1,2] = j\}$, and $\{N(2,3] = k\}$ are independent,

$$\begin{aligned} P(A) &= P(B_{020}) + P(B_{111}) + P(B_{202}) \\ &= e^{-\lambda} \cdot \left(\frac{\lambda^2}{2!} e^{-\lambda}\right) \cdot e^{-\lambda} + \left(\lambda e^{-\lambda}\right)^3 + \left(\frac{\lambda^2}{2!} e^{-\lambda}\right) \cdot e^{-\lambda} \cdot \left(\frac{\lambda^2}{2!} e^{-\lambda}\right) \\ &= \left(\frac{\lambda^2}{2} + \lambda^3 + \frac{\lambda^4}{4}\right) e^{-3\lambda}. \end{aligned}$$

(c) Find the probability that there are two arrivals in the interval $(1,2]$, given that there are two arrivals in the interval $(0,2]$ and two arrivals in the interval $(1,3]$.

Solution: Here, we are asked for the conditional probability

$$P\left(\{N(1,2] = 2\} \mid A\right) = \frac{P\left(\{N(1,2] = 2\} \cap A\right)}{P(A)}.$$

But $\{N(1,2] = 2\} \cap A = B_{020}$, and so

$$P\left(\{N(1,2] = 2\} \mid A\right) = \frac{P(B_{020})}{P(A)} = \frac{\frac{\lambda^2}{2}}{\frac{\lambda^2}{2} + \lambda^3 + \frac{\lambda^4}{4}} = \frac{2}{2 + 4\lambda + \lambda^2}.$$

2. **[Central limit theorem and Gaussian approximation]**

(a) A wafer contains 400 chips. Each is defective with probability 0.1, independently. Let X be the number of defective chips. Use the Gaussian approximation with continuity correction to estimate $P\{X \le 45\}$ and $P\{35 \le X \le 45\}$.

Solution: X is binomial with $n = 400$, $p = 0.1$, so $E[X] = np = 40$ and $\mathrm{Var}(X) = np(1-p) = 36$, giving $\sigma = 6$. Since X is integer valued, use the continuity correction:

$$P\{X \le 45\} \approx P\{\tilde{X} \le 45.5\} = \Phi\left(\frac{45.5 - 40}{6}\right) = \Phi(0.917) \approx 0.820,$$

$$P\{35 \le X \le 45\} \approx P\{34.5 \le \tilde{X} \le 45.5\} = \Phi(0.917) - \Phi(-0.917) = 2\Phi(0.917) - 1 \approx 0.641.$$

For comparison, the exact binomial values are 0.8214 and 0.6409. Without the continuity correction the first estimate would be $\Phi(5/6) \approx 0.798$, noticeably worse.

(b) An accountant adds 100 numbers, each first rounded to the nearest integer. The rounding errors are independent and uniformly distributed over $[-0.5, 0.5]$. Use the Gaussian approximation to estimate the probability that the total rounding error exceeds 5 in magnitude.

Solution: Let $S = E_1 + \cdots + E_{100}$ with each E_i uniform on $[-0.5, 0.5]$. From Section 3.3, $E[E_i] = 0$ and $\mathrm{Var}(E_i) = 1/12$. By independence, $E[S] = 0$ and $\mathrm{Var}(S) = 100/12 \approx 8.333$, so $\sigma_S \approx 2.887$. S is continuous type, so no continuity correction is needed:

$$P\{|S| > 5\} \approx P\{|\tilde{S}| > 5\} = 2Q\left(\frac{5}{2.887}\right) = 2Q(1.732) \approx 2(0.0416) = 0.0833.$$

The individual errors are small compared with the sum, exactly the setting where the CLT applies. Note that the worst case $|S| = 50$ is far larger than typical values.

(c) Packets arrive at a router according to a Poisson process with rate $\lambda = 3$ packets per millisecond (ms). Let T_{100} be the arrival time of the 100th packet. Find $E[T_{100}]$ and $\mathrm{Var}(T_{100})$, and use the Gaussian approximation to estimate $P\{T_{100} > 40\}$.
Solution: $T_{100} = U_1 + \cdots + U_{100}$ is Erlang with $r = 100$, $\lambda = 3$:

$$E[T_{100}] = \frac{r}{\lambda} = \frac{100}{3} \approx 33.33 \text{ ms}, \qquad \mathrm{Var}(T_{100}) = \frac{r}{\lambda^2} = \frac{100}{9} \approx 11.11 \text{ ms}^2, \qquad \sigma \approx 3.333 \text{ ms}.$$

$$P\{T_{100} > 40\} \approx Q\left(\frac{40 - 33.33}{3.333}\right) = Q(2) \approx 0.0228.$$

(d) For part (c), write the exact probability $P\{T_{100} > 40\}$ as a Poisson sum. The exact value is 0.0279. Briefly explain why the Gaussian approximation is less accurate here than in part (a). **Solution:** Using $\{T_r > t\} = \{N_t \le r-1\}$ with $N_{40} \sim Poi(3 \times 40) = Poi(120)$:

$$P\{T_{100} > 40\} = P\{N_{40} \le 99\} = \sum_{k=0}^{99} \frac{e^{-120}\, 120^k}{k!} \approx 0.0279.$$

The Gaussian estimate 0.0228 is about 18% too low. First, each exponential summand is strongly skewed to the right, so the Erlang pdf has a heavier right tail than any Gaussian, and 100 terms do not fully remove the skew. Second, the probability being estimated is a tail probability close to zero, where the Gaussian approximation is least accurate (Section 3.6.3). In part (a) the probabilities are moderate and the continuity correction is used, so the error is under 0.002.

3. **[ML estimation for a Gaussian distribution]**
The random variable X is assumed to have a Gaussian distribution. It is observed that $X = \sqrt{2}$.

(a) Suppose the variance of X is assumed to be one. Find the ML estimate of its mean given the observation.
Solution: The likelihood $f_X(\sqrt{2})$, or

$$\frac{1}{\sqrt{2\pi}}\, e^{-\frac{(\sqrt{2}-\mu)^2}{2}},$$

is maximized with respect to μ when $(\sqrt{2} - \mu)^2 = 0$, i.e., when the mean μ is equal to the observed value. So

$$\widehat{\mu}_{ML} = \sqrt{2}.$$

(b) (For this part we assume the mean is known and the variance is to be estimated, for the same observation $X = \sqrt{2}$.) Suppose the mean of X is assumed to be zero. Find the ML estimate of the variance of X given the observation.
Solution: Since $\frac{(\sqrt{2}-0)^2}{2\sigma^2} = \frac{1}{\sigma^2}$, the likelihood function $f_X(\sqrt{2})$ becomes

$$\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{\sigma^2}}.$$

We need to find the value of σ^2 that maximizes it. The derivative of the logarithm of the likelihood is

$$\frac{d}{d\sigma^2}\left(\text{constant} - \frac{1}{2}\ln(\sigma^2) - \frac{1}{\sigma^2}\right) = -\frac{1}{2\sigma^2} + \frac{1}{\sigma^4} = \frac{2 - \sigma^2}{2\sigma^4},$$

which is positive if $\sigma^2 < 2$ and negative if $\sigma^2 > 2$. Thus,

$$\widehat{\sigma^2}_{ML} = 2.$$

4. **[Transformation of a uniform distribution]**
X denotes a uniform random variable with mean 1 and variance 3. Find the pdf of $Y = |X|$.
Solution: Suppose the support of X is $[a, b]$. Then $a+b = 2$ and $(b-a)^2 = 36$. This allows us to calculate $a = -2$ and $b = 4$. The pdf $f_Y(y)$ is naturally divided into two regions: $0 \le y \le 2$ and $4 \ge y > 2$:

$$f_Y(y) = \begin{cases} \frac{1}{3} & 0 \le y \le 2 \\ \frac{1}{6} & 2 < y \le 4 \\ 0 & y > 4 \end{cases}$$

Formal derivation is as follows:

For $y < 0$, the CDF $F_Y(y) = 0$. Further, for $y > 4$, the CDF $F_Y(y) = 1$.

For any $0 \le y \le 2$, the CDF $F_Y(y) = P(Y \le y) = P(-y \le -X \le y) = \frac{y}{3}$

For any $2 \le y \le 4$, the CDF $F_Y(y) = P(Y \le y) = P(X \le y) = \frac{y+2}{6}$. Now, $f_Y(y) = \frac{d}{dy}F_Y$ follows directly.

5. **[A binary quantizer with Laplacian input]**
Suppose X has pdf $f_X(u) = \frac{e^{-|u|}}{2}$ and $Y = g(X)$ where $g(u) = \alpha(\text{sign}(u)) = \begin{cases} \alpha & \text{if } u \ge 0 \\ -\alpha & \text{if } u < 0 \end{cases}$ for some constant α. So Y is the output of a binary quantizer with input X.

   (a) Describe the pdf or pmf of Y.

   (b) Find the mean square quantization error, $E[(X-Y)^2]$. Your answer should depend on α. (Hint: $\int_0^\infty u^k e^{-u} du = k!$ for nonnegative integers k.)

   (c) Find α to minimize the mean square quantization error.

   Solution:

   (a) Y takes values in $\{-\alpha, \alpha\}$; its pmf is

$$p_Y(u) = \begin{cases} P\{X \ge 0\} = 0.5 & \text{if } u = \alpha \\ P\{X < 0\} = 0.5 & \text{if } u = -\alpha \\ 0 & \text{else} \end{cases}$$

   (b) Note that $(X-Y)^2$ is a function of X, equal to $(X+\alpha)^2$ if $X < 0$ and $(X-\alpha)^2$ if $X \ge 0$. So using LOTUS and symmetry yields

$$\begin{aligned} E[(X-Y)^2] &= \int_{-\infty}^{0} (u+\alpha)^2 \frac{e^u}{2} du + \int_0^\infty (u-\alpha)^2 \frac{e^{-u}}{2} du \\ &= \int_0^\infty (u-\alpha)^2 e^{-u} du \\ &= \int_0^\infty (u^2 - 2\alpha u + \alpha^2) e^{-u} du \\ &= 2 - 2\alpha + \alpha^2 \end{aligned}$$

(c) The derivative of the mean square error with respect to α is $2(-1+\alpha)$, so the mean square error is decreasing in α for $\alpha < 1$ and increasing for $\alpha > 1$. So $\alpha = 1$ minimizes the mean square error. ALTERNATIVELY, the mean square error can be expressed as $1 + (1-\alpha)^2$, which makes it clear that $\alpha = 1$ minimizes it.

6. **[Function of a random variable]**
Let X have pdf $f_X(u) = \frac{1}{2u^2}$ for $|u| \geq 1$ and $f_X(u) = 0$ for $|u| < 1$. Let $Y = \sqrt{|X|}$.

(a) Using LOTUS, find $E[Y]$.

(b) Find the pdf of Y.

(c) (This part does not involve Y.) Find the nondecreasing function h so that $h(X)$ is uniformly distributed over $[0,1]$. Be as explicit as possible.

Solution:

(a) By LOTUS and symmetry,

$$\begin{aligned} E[Y] &= \int_{-\infty}^{\infty} \sqrt{|u|} f_X(u) du \\ &= \int_{-\infty}^{-1} \sqrt{-u} \frac{1}{2u^2} du + \int_1^{\infty} \sqrt{u} \frac{1}{2u^2} du \\ &= \int_1^{\infty} \sqrt{u} \frac{1}{u^2} du \\ &= \int_1^{\infty} u^{-3/2} du \\ &= (-2u^{-1/2})\Big|_1^{\infty} = 2 \end{aligned}$$

(b) Note that Y ranges over $[1, \infty)$. For $1 \leq c < \infty$,

$$F_Y(c) = P\{Y \leq c\} = 2P\{1 \leq X \leq c^2\} = \int_1^{c^2} \frac{1}{u^2} du = -\frac{1}{u}\Big|_1^{c^2} = 1 - \frac{1}{c^2}.$$

Differentiating with respect to c yields

$$f_Y(c) = \begin{cases} 2/c^3 & \text{if } c \geq 1 \\ 0 & \text{else.} \end{cases}$$

(c) By Example 3.8.11 of the notes, the function $h(u)$ is just the CDF of X: $h(c) = F_X(c)$. For $c \leq -1$,

$$F_X(c) = \int_{-\infty}^{c} \frac{1}{2u^2} du = -\frac{1}{2u}\Big|_{-\infty}^{c} = -\frac{1}{2c}.$$

In particular, $F_X(-1) = 0.5$. Since $f_X(u) = 0$ for $u \in (-1, 1)$, it follows that F_X is flat over the interval $[-1, 1]$, with $F_X(1) = 0.5$. For $c \geq 1$

$$F_X(c) = 0.5 + \int_1^c \frac{1}{2u^2} du = 0.5 + \left(-\frac{1}{2u}\Big|_1^c \right) = 1 - \frac{1}{2c}$$

Combining the above,

$$h(c) = F_X(c) = \begin{cases} -\frac{1}{2c} & \text{if } c \leq -1 \\ 0.5 & \text{if } -1 < c \leq 1 \\ 1 - \frac{1}{2c} & \text{if } c > 1. \end{cases}$$