

ECE 313: Problem Set 6

Due: Sunday, Oct 25 at 06:59:00 p.m.

Reading: *ECE 313 Course Notes, Sections 3.3-3.6.2*

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:

NAME

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PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Uniform distribution]**

Let X be uniformly distributed over an interval $[a, b]$ with $E[X] = 4$ and $\text{Var}(X) = 3$.

- (a) Find a and b , and write the pdf f_X .
- (b) Find $P\{X > 5\}$.
- (c) Find $P(X > 5 \mid X > 3)$. Is the conditional distribution of X given $X > 3$ still uniform? Over what interval?
- (d) Find the value c such that $P\{X \leq c\} = 0.75$.
- (e) A stick of unit length is broken at a point U , uniformly distributed over $[0, 1]$. Find the probability that the longer piece is at least twice as long as the shorter piece. Also find the expected length of the longer piece.

2. **[Exponential distribution, memoryless property]**

The lifetime T (in hours) of a laser diode is exponentially distributed with parameter λ . Its median lifetime is 2000 hours, meaning $P\{T \leq 2000\} = 1/2$.

- (a) Find the parameter λ .
- (b) Find the mean and standard deviation of T .
- (c) Find $P\{T > 3000\}$. Express it in the simplest form.
- (d) A diode has already operated for 1500 hours without failing. Find the probability it is still working at hour 4000.
- (e) The diode is inspected only at the end of each hour, so failure is detected at hour $K = \lceil T \rceil$ (T rounded up to an integer). Show that K is geometrically distributed and find its parameter p .

3. **[Poisson processes]**

Packets arrive at a router according to a Poisson process with rate $\lambda = 3$ packets per millisecond (ms). Let N_t be the number of arrivals in $[0, t]$, with t in ms.

- (a) Find the probability that no packets arrive in the first 0.5 ms.
- (b) Find the probability that exactly 4 packets arrive in $(0, 1]$ and exactly 2 packets arrive in $(1, 1.5]$.
- (c) Find $P(N_2 = 6 \mid N_1 = 4)$.
- (d) Find $P(N_1 = 2 \mid N_2 = 6)$. Identify the distribution family and its parameters.
- (e) Given that exactly one packet arrived during $[0, 1]$, find the conditional probability that it arrived during $[0, 0.25]$.

4. **[Erlang distribution]**

A satellite subsystem uses one active component plus two cold spares. When the active component fails, a spare is switched in instantly (spares do not fail while unused). Component lifetimes are independent and exponentially distributed with mean 100 hours. Let T be the time at which the third (last) component fails.

- (a) Identify the distribution of T and write its pdf.
- (b) Find $E[T]$, $\text{Var}(T)$, and the standard deviation of T .
- (c) Find $P\{T > 200\}$.
- (d) Find $P\{T \leq 500\}$.
- (e) Find the mode of T (the value of t that maximizes $f_T(t)$).
- (f) Suppose the designer may use r components in total (one active plus $r - 1$ spares). Find the smallest r such that $P\{T_r > 200\} \geq 0.95$, where T_r is the time at which the r -th component fails.

5. **[Scaling rule for pdfs]**

Let X have pdf $f_X(u) = 2u$ for $0 \leq u \leq 1$ and $f_X(u) = 0$ otherwise.

- (a) Let $Y = 3X - 1$. Use the scaling rule to find f_Y . Sketch f_X and f_Y , and check that f_Y integrates to one.
- (b) Find $E[Y]$ and $\text{Var}(Y)$ two ways: from f_Y directly, and from $E[X]$, $\text{Var}(X)$, and the linear scaling relations.
- (c) Let $W = -2X + 1$. The scaling rule (3.5) requires $a > 0$, so derive f_W directly from the CDF.
- (d) A server's response time T , in milliseconds, is exponentially distributed with mean 250 ms. Let S be the same response time in seconds. Find the pdf of S and identify its distribution.
- (e) More generally, let T_r have the Erlang distribution with parameters r and λ , and let $c > 0$. Use the scaling rule to show that cT_r has the Erlang distribution with parameters r and λ/c .

6. **[Gaussian distribution, Q and Φ functions]**

Let X have the $N(5, 16)$ distribution. Express each answer using Φ or Q , then give a numerical value.

- (a) $P\{X > 11\}$
- (b) $P\{1 < X \leq 13\}$
- (c) $P\{|X - 5| > 8\}$
- (d) $P\{X^2 > 81\}$
- (e) Find c such that $P\{X > c\} = 0.05$.
- (f) Find a such that $P\{|X - 5| \leq a\} = 0.99$.