

ECE 313: Problem Set 5: Problems and Solutions

Due: Friday, Oct. 9 at 06:59:00 p.m.

Reading: *ECE 313 Course Notes, Section 2.11, 3.1.*

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:

NAME

NETID

SECTION

PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Detection Problem with the geometric distribution]**

The number of attempts, Y , required for a certain basketball player to make a 25 foot shot, is observed, in order to choose one of the following two hypotheses:

H_1 (outstanding player) : Y has the geometric distribution with parameter $p = 0.6$,

H_0 (average player) : Y has the geometric distribution with parameter $p = 0.25$.

(a) Describe the ML decision rule. Express it as directly in terms of Y as possible.

Solution: The ML rule is to declare H_1 if $\Lambda(Y) \geq 1$, where Λ is the likelihood ratio function, defined by

$$\Lambda(k) = \frac{p_1(k)}{p_0(k)} = \frac{(0.6)(0.4)^{k-1}}{(0.25)(0.75)^{k-1}} = (2.4) \left(\frac{8}{15} \right)^{k-1}, \quad k \geq 1.$$

Note that $\Lambda(k)$ is decreasing in k with

$$\Lambda(1) = 2.4, \Lambda(2) = 1.28, \Lambda(3) \approx 0.683$$

Therefore, the ML decision rule declares H_1 if $Y \leq 2$ and declares H_0 otherwise.

(b) Find $p_{\text{false alarm}}$ and p_{miss} for the ML rule.

Solution: The probability of false alarm is

$$P_{\text{false alarm}} = P(Y \leq 2 \mid H_0) = p_0(1) + p_0(2) = 0.25 + (0.25)(0.75) = 0.4375.$$

The probability of miss is

$$P_{\text{miss}} = P(Y \geq 3 \mid H_1) = \sum_{k=3}^{\infty} (0.6)(0.4)^{k-1} = (0.6)(0.4)^2 (1 + 0.4 + (0.4)^2 + \dots) = \frac{(0.6)(0.4)^2}{1 - 0.4} = 0.16,$$

where we used the formula for the sum of a geometric series.

- (c) Describe the MAP decision rule under the assumption that H_0 is a priori twice as likely as H_1 . Express the rule as directly in terms of Y as possible.

Solution: The prior probabilities are $\pi_0 = P(H_0) = 2/3$ and $\pi_1 = P(H_1) = 1/3$. The MAP rule declares H_1 if

$$\Lambda(Y) \geq \frac{\pi_0}{\pi_1} = 2.$$

Using the likelihood ratio from part (a),

$$(2.4) \left(\frac{8}{15} \right)^{Y-1} \geq 2,$$

which is equivalent to

$$Y \leq 1 + \frac{\ln(1.2)}{\ln(15/8)} \approx 1.29.$$

Since $\Lambda(1) = 2.4 \geq 2$ and $\Lambda(2) = 1.28 < 2$, the MAP decision rule is

decide H_1 if $Y = 1$,

and decide H_0 otherwise. That is, the MAP rule declares H_1 if and only if the player is successful on the first attempt.

- (d) Find the average error probability, p_e , for both the ML rule and the MAP rule, using the same prior distribution given in part (c). For which rule is the average error probability smaller?

Solution: For the MAP rule,

$$P_{\text{false alarm}} = P(Y = 1 \mid H_0) = p_0(1) = 0.25,$$

and

$$P_{\text{miss}} = P(Y \geq 2 \mid H_1) = 1 - p_1(1) = 0.4.$$

Hence,

$$P_e^{MAP} = \pi_0 P_{\text{false alarm}} + \pi_1 P_{\text{miss}} = \frac{2}{3}(0.25) + \frac{1}{3}(0.4) = 0.3.$$

For the ML rule, using part (b),

$$P_e^{ML} = \pi_0 P_{\text{false alarm}} + \pi_1 P_{\text{miss}} = \frac{2}{3}(0.4375) + \frac{1}{3}(0.16) = 0.345.$$

Thus, the MAP rule has the smaller average error probability. This is expected, since the MAP rule based on a given prior distribution minimizes the average error probability for that prior distribution.

2. [Testing hypotheses about a die]

Consider a binary hypothesis testing problem based on observation of n independent rolls of a die. Let $X_1, \dots, X_n$ denote the numbers rolled on the die. Let a denote a known constant that is slightly greater than one. The two hypotheses are:

H_0 : the die is fair

H_1 : for each roll of the die, i shows with probability $p_i = Ca^i$, where $C = \frac{1-a}{a-a^7}$, so that the probabilities sum to one.

- (a) Find simple expressions for $p_i(u_1, \dots, u_n) = P(X_1 = u_1, \dots, X_n = u_n \mid H_i)$ for $i = 0$ and $i = 1$, where $u_i \in \{1, 2, 3, 4, 5, 6\}$ for each i . Express your answers using the variables t_k , for $1 \leq k \leq 6$, where t_k is the number of the n rolls that show k . The vector $(t_1, \dots, t_6)$ is called the type vector of $(u_1, \dots, u_n)$. Intuitively, the order of the observations shouldn't matter, so decision rules will naturally only depend on the type vector of the observation sequence.

Solution: By the independence of the rolls,

$$p_i(u_1, \dots, u_n) = P(X_1 = u_1 \mid H_i) \cdots P(X_n = u_n \mid H_i).$$

For $i = 0$, $P(X_m = u_m \mid H_0) = \frac{1}{6}$ for any choice of u_m , so

$$p_0(u_1, \dots, u_n) = 6^{-n}.$$

For $i = 1$, the factor Ca^k appears t_k times for each $1 \leq k \leq 6$, so

$$p_1(u_1, \dots, u_n) = (Ca)^{t_1} (Ca^2)^{t_2} (Ca^3)^{t_3} \cdots (Ca^6)^{t_6} = C^{t_1 + \cdots + t_6} a^{t_1 + 2t_2 + \cdots + 6t_6} = C^n a^s,$$

where s is the sum of the rolls:

$$s = t_1 + 2t_2 + \cdots + 6t_6 = u_1 + \cdots + u_n.$$

- (b) Find a simple expression for the likelihood ratio, $\Lambda(u_1, \dots, u_n) = \frac{p_1(u_1, \dots, u_n)}{p_0(u_1, \dots, u_n)}$, and describe, as simply as possible, the likelihood ratio test for H_1 vs. H_0 given the observations $(u_1, \dots, u_n)$.

Solution: By part (a),

$$\Lambda(u_1, \dots, u_n) = \frac{C^n a^s}{6^{-n}} = (6C)^n a^s.$$

The likelihood ratio test with threshold τ decides H_1 if

$$(6C)^n a^s \geq \tau.$$

Taking logarithms and using $\ln a > 0$ (since $a > 1$), this is equivalent to

$$s \geq \gamma, \quad \text{where} \quad \gamma = \frac{\ln \tau - n \ln(6C)}{\ln a}.$$

That is, the test decides H_1 if the sum of the numbers rolled is at least γ . (For the ML rule, $\tau = 1$.)

- (c) In particular, suppose that $n = 100$, $(t_1, \dots, t_6) = (18, 12, 13, 19, 18, 20)$, and $a = 1.1$. Which hypothesis does the maximum likelihood decision rule select?

Solution: With $a = 1.1$,

$$C = \frac{1 - 1.1}{1.1 - (1.1)^7} \approx 0.1178,$$

so $6C \approx 0.7069$. For the ML rule, $\tau = 1$, so

$$\gamma_{ML} = \frac{-100 \ln(6C)}{\ln(1.1)} \approx \frac{34.68}{0.09531} \approx 363.86.$$

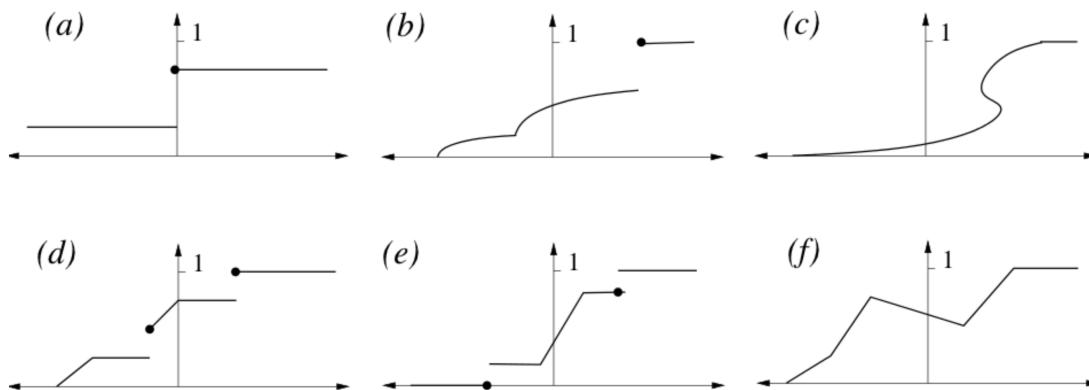
Therefore, the ML rule decides H_1 if $s \geq 364$ and decides H_0 otherwise. For the given data,

$$s = 1(18) + 2(12) + 3(13) + 4(19) + 5(18) + 6(20) = 367.$$

Since $367 \geq 364$, the ML rule decides that H_1 is true.

3. [CDFs]

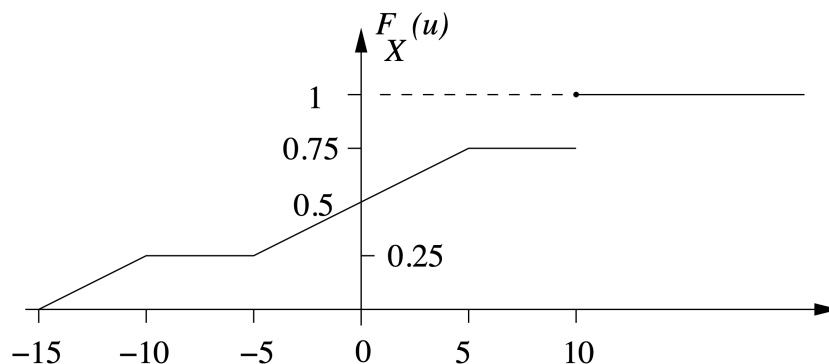
Which of the six plots below show valid CDFs? For each one that is not valid, state a property of CDFs that is violated.



Solution: The functions shown in plots (b) and (d) are valid CDFs and the other four are not. The function in (a) does not converge to zero at $-\infty$ nor to one at $+\infty$. Plot (c) doesn't even show a function—for some values of u there are two values of $F(u)$ shown. The function in (e) is not right continuous. The function in (f) is not nondecreasing.

4. [CDFs]

Let X be a random variable with the CDF shown.



Compute the following probabilities:

(a) $P\{X \leq -12\}$

Solution: On the interval $[-15, -10]$, the CDF increases linearly from 0 to 0.25, so $F_X(-12) = \frac{3}{5} \cdot 0.25 = 0.15$. Hence, $P\{X \leq -12\} = F_X(-12) = 0.15$.

(b) $P\{-12 < X < 4\}$

Solution: Since there are no jumps at -12 or 4 , $P\{-12 < X < 4\} = F_X(4) - F_X(-12)$. From the graph, on the interval $[-5, 5]$, the CDF increases linearly from 0.25 to 0.75, so $F_X(4) = 0.25 + \frac{9}{10} \cdot 0.5 = 0.7$. Also, $F_X(-12) = 0.15$. Therefore, $P\{-12 < X < 4\} = 0.7 - 0.15 = 0.55$.

(c) $P\{X > 5\}$

Solution: Since $F_X(5) = 0.75$, $P\{X > 5\} = 1 - F_X(5) = 1 - 0.75 = 0.25$.

(d) $P\{X \geq 10\}$

Solution: There is a jump at $x = 10$, so $P\{X \geq 10\} = 1 - P\{X < 10\} = 1 - F_X(10^-) = 1 - 0.75 = 0.25$.

(e) $P\{|X| \leq 5\}$

Solution: The event $\{|X| \leq 5\}$ is the same as $\{-5 \leq X \leq 5\}$. Thus, $P\{|X| \leq 5\} = F_X(5) - F_X(-5^-) = 0.75 - 0.25 = 0.5$.

(f) $P\{X^2 \geq 100\}$

Solution: The event $\{X^2 \geq 100\}$ is the same as $\{X \leq -10\} \cup \{X \geq 10\}$. These two events are disjoint, so $P\{X^2 \geq 100\} = P\{X \leq -10\} + P\{X \geq 10\}$. From the graph, $P\{X \leq -10\} = F_X(-10) = 0.25$ and $P\{X \geq 10\} = 0.25$. Therefore, $P\{X^2 \geq 100\} = 0.25 + 0.25 = 0.5$.