ECE 313: Problem Set 1: Problems and Solutions

Due: Friday, September 11 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Chapter 1

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.
Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned page numbers.**

1. **[Sample Space]**
   Consider events A and B defined on a sample space. If the probability that at least one of the two events occurred is 0.7 and the probability that at least one of the events did *not* occur is 0.9, what is the probability that *exactly one* of the two events occurred?

   Solution: One way to solve this problem is through the use of Karnaugh maps. Below is a solution that can be reduced to the aforementioned approach, but which does not make explicit reference to the maps.
   Let $P(AB) = x, P(AB^c) = y, P(A^cB) = z, P(A^cB^c) = w$. We are given that, probability of at least one of the two events happening $\{AB, AB^c, A^cB\}$ is 0.7.

   $$x + y + z = 0.7 \tag{1}$$

   The probability of at least one of the two events did not occur $\{AB^c, A^cB, A^cB^c\}$ is 0.9.

   $$y + z + w = 0.9 \tag{2}$$

   Since the sum of probabilities of all the events is 1,

   $$x + y + z + w = 1 \Leftrightarrow x + w = 1 - (y + z) \tag{3}$$

   We need to find the probability that exactly one of the two events occurred $\{AB^c, A^cB\}$ i.e.,$y + z$. From (1),(2), and (3)

   $$\begin{aligned} 2(y + z) + x + w &= 0.9 + 0.7 \\ \Leftrightarrow 2(y + z) + 1 - (y + z) &= 1.6 \\ \Leftrightarrow y + z &= 0.6 \end{aligned}$$

2. **[Bits]**
An experiment consists of observing the contents of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.

(a) Let A denote the event that the least significant bit is a ONE. What is $P(A)$?
Solution: Any arbitrary 7-digit binary number concatenated with a ONE is a member of event A. Thus, there are $2^7 = 128$ such numbers out of 256 8-digit numbers. Since all numbers are equi-probable,

$$P\{A\} = 128 \cdot 2^{-8} = \frac{1}{2}$$

*A more mind-boggling proof: First, we define an operation called "flip" that operates on a single register: flip changes ZERO to a ONE, and a ONE to a ZERO. Second, note that all 8-digit binary numbers must end in either ZERO or ONE. Finally, for every unique 8-digit binary number that ends with a ONE, flip each bit of this number, and you would obtain a unique 8-digit number that ends with a ZERO. The opposite is true by symmetry, thus $|A| = |A^c|$. Since it is given that all 8-digit numbers are equally likely: $P\{A\} = P\{A^c\}$, and $P\{A\} = \frac{1}{2} = 128 \cdot 2^{-8}$. This second proof can be easily modified so that instead of ones-complements, we use odd-even pairs to arrive at the answer.

(b) Let B denote the event that the register contains 5 ONEs and 3 ZEROs. What is $P(B)$?
Solution: $P\{B\} = \binom{8}{5}(\frac{1}{2})^5(\frac{1}{2})^3$.
The first term indicates the number of different combinations that satisfy the condition that 5 out of 8 digits are ONEs. The second term indicates the probability that these 5 digits are ONEs. The third term indicates the probability that the remaining 3 digits are ZEROs.

$$\begin{aligned} P\{B\} &= \binom{8}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3 \\ &= 56 \cdot 2^{-8} \end{aligned}$$

(c) What is $P(A \cup B)$? What is $P(A \cap B)$? What is the probability that exactly one of A and B occur, i.e. what is $P(A \oplus B)$?
Solution: The intersection of event A and B can be described as: *Out of the first 7 digits, there are 4 ONEs and 3 ZEROs; and the last digit must be a ONE.* Thus, the probability of the intersection is:

$$\begin{aligned} P\{A \cap B\} &= \binom{7}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) \\ &= 35 \cdot 2^{-8} \end{aligned}$$

Calculating the other quantities is then straight forward:

$$\begin{aligned} P\{A \cup B\} &= P\{A\} + P\{B\} - P\{A \cap B\} \\ &= 149 \cdot 2^{-8} \\ P\{A \oplus B\} &= P\{A\} + P\{B\} - 2P\{A \cap B\} \\ &= 114 \cdot 2^{-8} \end{aligned}$$

(d) Let C denote the event that there are 4 ONEs and 4 ZEROs in the register. What is $P(C)$?

Solution: Similar to part b,

$$P\{C\} = \binom{8}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4 = 70 \cdot 2^{-8}$$

3. **[Subsets]**
Find $P(A \cup (B^c \cup C^c)^c)$ in each of the following four cases:

(a) A, B, and C are disjoint (mutually exclusive) events and $P(A) = 1/3$.
Solution: $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C))$ by DeMorgan's theorem.
$B \cap C = \emptyset$ and hence $P(A \cup (B \cap C)) = P(A \cup \emptyset) = P(A) = 1/3$.

(b) $P(A) = 2P(B \cap C) = 4P(A \cap B \cap C) = 1/2$.
Solution: $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C))$ by DeMorgan's theorem.
$P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C))$
$= 4P(A \cap B \cap C) + 2P(A \cap B \cap C) - P(A \cap B \cap C) = 5P(A \cap B \cap C) = 5/8$.

(c) $P(A) = 1/2$, $P(B \cap C) = 1/3$, and $P(A \cap C) = 0$.
Solution: $P(A \cup (B^c \cup C^c)^c) = P(A \cup (B \cap C))$ by DeMorgan's theorem.
$P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = 1/2 + 1/3 - 0 = 5/6$.

(d) $P(A^c \cap (B^c \cup C^c)) = 0.6$.
Solution: $(A \cup (B^c \cup C^c)^c)^c = A^c \cap (B^c \cup C^c)$. Hence, $P(A \cup (B^c \cup C^c)^c) = 1 - 0.6 = 0.4$.

4. **[Cards]**
Suppose five cards are drawn from a standard 52 card deck of playing cards, as described in Example 1.4.3, with all possibilities being equally likely.

(a) $E1$ is the event that all five cards are one of A, K, Q, J (though not necessarily of the same suit). Find $P(E1)$.
Solution: There are $\binom{16}{5} = 4368$ ways of cards in the event $E1$. Normalizing by the number of all outcomes $\binom{52}{5}$, we get the probability of $E1$ to be $P(E1) = \frac{\binom{16}{5}}{\binom{52}{5}}$.

(b) $E2$ is the event that either all the five cards are from the same suit or four of the five cards are A, K, Q, J from a common suit. Find $P(E2)$.
Solution: Now $E2$ is the union of two events, $E4$ and $E3$, which are defined as follows:

- $E3$ is the event of having four of the cards to be A, K, Q, J of the same suit;
- $E4$ is the event of having all five cards of the same suit.

The number of outcomes in $E3$ is $4 \cdot 48 = 192$(because there are four suits to choose from for the A, K, Q, J and the fifth card can be any of the remaining 48 cards).The number of outcomes in $E4$ is $\binom{13}{5} \cdot 4 = 5148$. The number of outcomes common to $E3$ and $E4$ is $4 \cdot 9 = 36$. So we conclude that the number of outcomes in $E2 = E4 \cup E3$ is $5148 + 192 - 36 = 5304$. Therefore, $P(E2) = \frac{5304}{\binom{52}{5}}$.

5. **[Counting rectangles on a chessboard]**
Consider a 6x6 chessboard, which consists of 36 squares in 6 rows and 6 columns.

   (a) How many different rectangles, comprised entirely of chessboard squares, can be drawn on the chessboard? Hint: there are 7 horizontal and 7 vertical lines in the chessboard.

   Solution: A rectangle is uniquely described by the pair of horizontal lines and the pair of vertical lines that form its sides. Since there are $\binom{7}{2} = \frac{7\times6}{1\times2} = 21$ choices for the pair of horizontal lines and, similarly, 21 choices for the pair of vertical lines, there are $\binom{7}{2}^2 = 21 \times 21 = 441$ rectangles.

   (b) One of the rectangles you counted in part (a) is chosen at random. What is the probability that it is square shaped?

   Solution: Number the horizontal and vertical lines as 0 through 6. Then, for any given k, $1 \le k \le 6$, we get a $k \times k$ square if we pick the pair of horizontal lines as one of $(0,k),(1,k+1),(2,k+2),\ldots,(6-k,6)$ and similarly for the pair of vertical lines. Thus, the number of $k \times k$ squares is $(7-k)^2$. Hence, the number of square shaped rectangles is $1^2+2^2+3^2+4^2+5^2+6^2 = 91$. So the probability of getting a square shaped rectangle is $\frac{91}{21\times21} = \frac{13}{63}$.

6. **[First and second moments of a ternary random variable]**
This problem focuses on the possible mean and variance of a random variable X with support set $\{-1,0,1\}$. Let $p_X(-1) = a$ and $p_X(1) = b$ and $p_X(0) = 1-a-b$. Let $\mu = E[X]$, $\sigma^2 = \mathrm{Var}(X)$, and $m_2 = E[X^2]$.

   (a) What are the conditions for a and b to make the pmf valid?

   Solution: The pmf is non-negative, meaning

$$p_X(-1) = a \ge 0, \quad p_X(0) = 1-a-b \ge 0, \quad p_X(1) = b \ge 0$$

$$\implies a \ge 0,\ b \ge 0,\ a+b \le 1.$$

   (b) Find (a,b) so that $\mu = \frac{1}{2}$ and $\sigma = \frac{1}{2}$.

   Solution: Since $m_2 = \sigma^2 + \mu^2$, the given requirement is equivalent to $\mu = 1/2$ and $m_2 = 1/2$. In general, μ and m_2 can be expressed in terms of a and b as

$$\mu = b - a, \qquad m_2 = a + b.$$

   Applying the given constraints we have

$$\mu = b - a = 1/2, \qquad m_2 = a + b = 1/2 \implies a = 0,\ b = 1/2.$$

   Note that this choice for (a,b) is valid since the conditions $a \ge 0$, $b \ge 0$, and $a+b \le 1$ hold.

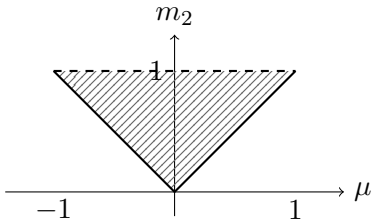
   (c) Express a and b in terms of μ and m_2. Determine and sketch the region of (μ, m_2) pairs for which there is a valid choice of (a,b).

   Solution: Since $\mu = b - a$ and $m_2 = a + b$, $a = \frac{m_2-\mu}{2}$ and $b = \frac{m_2+\mu}{2}$.

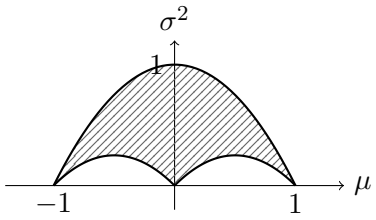
$$a \ge 0 \implies \mu \le m_2$$

$$b \ge 0 \implies \mu \ge -m_2 \implies |\mu| \le m_2$$

$$a + b \le 1 \implies m_2 \le 1$$



(d) Determine and sketch the set of (μ, σ^2) pairs for which there is a valid choice of (a, b).

Solution: From the support set, we know that the mean μ must be in the range $-1 \leq \mu \leq 1$. Since $m_2 \leq 1$, we have $\sigma^2 = m_2 - \mu^2 \leq 1 - \mu^2$. Then

$$m_2 \geq |\mu| \implies \sigma^2 \geq |\mu| - \mu^2 = \begin{cases} \frac{1}{4} - \left(\mu - \frac{1}{2}\right)^2, & \mu \geq 0 \\ \frac{1}{4} - \left(\mu + \frac{1}{2}\right)^2, & \mu < 0 \end{cases}$$