Agenda

Final Review I (Post-midterm2)

- BHT
- Joint Probability
- Estimators
- Functions of RV

Binary Hypothesis Testing

- Likelihood matrix/ function
  - $f_i(k) = P\{X = k|H_i\}$
  - Likelihood Ratio $\Lambda(k) = \frac{f_1(k)}{f_0(k)}$
  - ML method - $f_1(k)\ vs.\ f_0(k)$ or $\Lambda(k) > 1$?

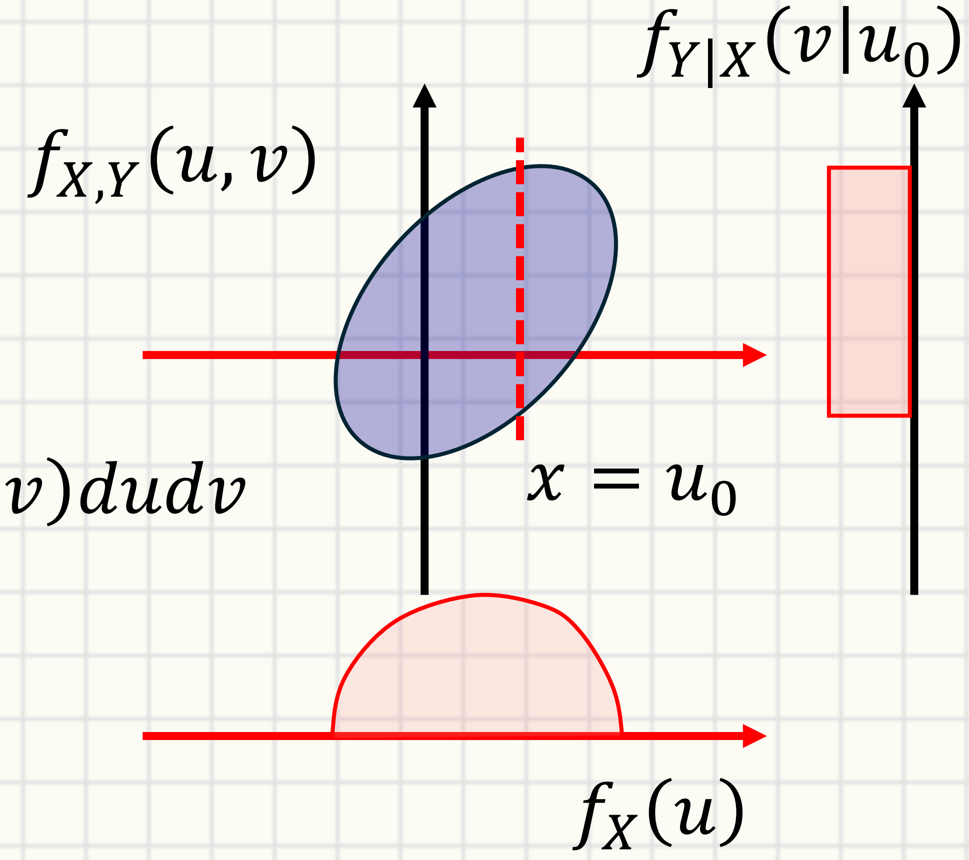
- Joint Probability Matrix $P(X_i H_i)$
  - MAP method - $\Lambda(k) > \tau_{MAP} = \frac{\pi_0}{\pi_1}$

Example -

Let $f_1 \sim N(\mu = 1, \sigma^2 = 2)$ and $f_0 \sim N(\mu = 0, \sigma^2 = 1)$

- Find ML rule and MAP rule corresponding $P_{miss}, P_{false\ alarm}$

Joint Probability Density



X and Y are described by $f_{X,Y}(u,v)$

- $E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\,du\,dv$
  - Double integrate over x and y
- $f_{Y|X}(v|u_0) = \frac{f_{X,Y}(u_0,v)}{f_X(u_0)}$
  - Conditional distribution
  - Useful in **unconstrained estimator**
- $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$ and $E[X] = \int_{-\infty}^{\infty} u \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\,du$
  - Marginalization

Example

$$f_{X,Y}(u,v) = \begin{cases} A(1-|u-v|) & if \quad 0 < u < 1, 0 < v < 1 \\ 0 & else \end{cases}$$

- Solve A
- Find $f_X(u)$ and $f_Y(v)$
- Find $P\{X + Y < 1 | X > \frac{1}{2}\}$

Estimators

- Constant estimator
  - $c^* = E[Y]$, MSE = $Var(Y)$

- Unconstraint estimator
  - $g^*(X) = E[Y|X]$ MSE=$E[Y^2] - E[(E[Y|X])^2] = E[Y^2] - E[\left(g^*(X)\right)^2]$
  - Best estimator, but requires $f_{Y|X}$

- Linear estimator
  - $\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y(\frac{X-\mu_X}{\sigma_X})$
  - MSE= $\sigma_Y^2 - \frac{\left(Cov(X,Y)\right)^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$

Example

$$f_{X,Y}(u,v) = \begin{cases} c(u+v) & if \quad 0 < u < 1, 0 < v < 1 \\ 0 & else \end{cases}$$

- Find c^* for Y

- Find $E[Y|X]$

- Find $\hat{E}[Y|X]$

Functions of RVs

- Single RV $Y = g(X)$
  1. Find support of X and Y
  2. Find $F_Y(c)$ from integrating $f_X(x)$ over $\{x: g(x) \leq c\}$
  3. Get $f_Y = F_Y{}'$

- Multiple RV $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
  - $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v) = \frac{1}{|\det(A)|} f_{X,Y}(A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix})$

Example

$W = 2X - Y, Z = X + 2Y$. Express $f_{W,Z}(\alpha, \beta)$ in terms of $f_{X,Y}$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$

- $Det(A) =$

- For $(W, Z) = (\alpha, \beta)$, $(X, Y) =$

- $f_{W,Z}(\alpha, \beta) =$