Last lecture

Minimum mean square error estimation (Ch 4.9)

- Recap
  - Constant estimators
  - Unconstrained estimators
  - Linear estimators
- Examples

Agenda

Joint Gaussian Distribution (Ch 4.11)

- Motivation
- Facts
- Examples

Joint Gaussian distribution

Motivation

Describe the joint distribution of (Height, Weight)=(X, Y) of the class

- Metrics: $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{X,Y}$
- Is $aX + bY$ Gaussian?

Def. 4.11.1 RV X and Y are said to be **jointly Gaussian** if every linear combination $aX + bY$ is a Gaussian random variable

- $f_{X,Y}(u,v) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} exp(-\frac{\left(\frac{u-\mu_X}{\sigma_X}\right)^2 + \left(\frac{v-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{u-\mu_X}{\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right)}{2\sqrt{1-\rho^2}})$

Justifying a bivariate normal

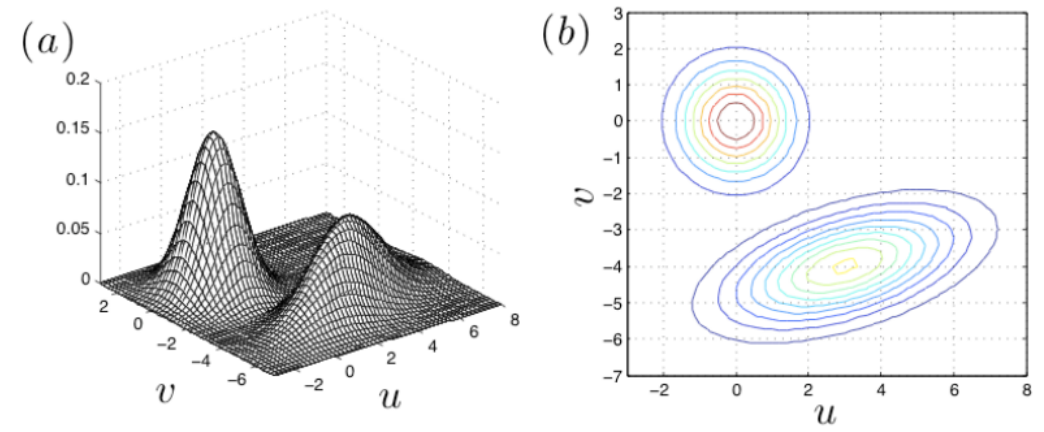
A bivariate normal $f_{X,Y}(u,v) = C \times exp\big(-P(u,v)\big)$

- $P(u,v) = au^2 + buv + cv^2 + du + ev + f$
- $P(u,v) \to \infty$ as $|u| + |v| \to \infty$
- $a, c > 0, b^2 - 4ac < 0$

Standard 2-d Normal to Bivariate Normal

Let W, Z be independent standard normal

- $f_{W,Z}(\alpha, \beta) = \frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{b^2}{2}}}{\sqrt{2\pi}} = \frac{e^{-\frac{\alpha^2+\beta^2}{2}}}{2\pi}$



- $\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} W \\ Z \end{pmatrix} + \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$

- $A = \begin{bmatrix} \sqrt{\frac{\sigma_X^2(1+\rho)}{2}} & -\sqrt{\frac{\sigma_X^2(1-\rho)}{2}} \\ \sqrt{\frac{\sigma_Y^2(1+\rho)}{2}} & \sqrt{\frac{\sigma_Y^2(1-\rho)}{2}} \end{bmatrix}$

Facts

If X and Y forms the bivariate normal with $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$

- $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$

- $aX + bY$ is a Gaussian for any $a, b \in \mathbb{R}$

- $\rho = \rho_{X,Y}$, X and Y are independent iif $\rho = 0$

- $E[Y|X] = \hat{E}[Y|X]$

- $(Y|X = u) \sim N(\hat{E}[Y|X = u], \sigma_e^2)$

Example

Let X and Y be jointly Gaussian with mean $(0,0)$. $\sigma_X^2 = 5$, $\sigma_Y^2 = 2$ and $Cov(X,Y) = -1$. Find $P\{X + 2Y \geq 1\}$ in terms of Φ function

- $Z = X + 2Y$ is Gaussian
- $\mu_Z =$
- $Var(Z) =$

Example

Let X and Y be jointly Gaussian RV with mean 0, variance 1, and $Cov(X, Y) = \rho$. Find $E[Y^2|X]$

- $\hat{E}[Y|X] =$
- For any RV, $Var(Z) = E[Z^2] - (E[Z])^2$

Example

Let X and Y be jointly Gaussian RV with mean 0, variance 1, and $Cov(X, Y) = 0.5$. Find

- $Var(3X - 2Y)$
- $P\{(3X - 2Y)^2 \leq 28\}$ in terms of Φ
- $E[Y|X = 3]$