Last lecture

Correlation and covariance (Ch 4.8)

- Examples

Minimum mean square error estimation (Ch 4.9)

- Constant estimators
- Unconstrained estimators
- Linear estimators

Agenda

Minimum mean square error estimation (Ch 4.9)

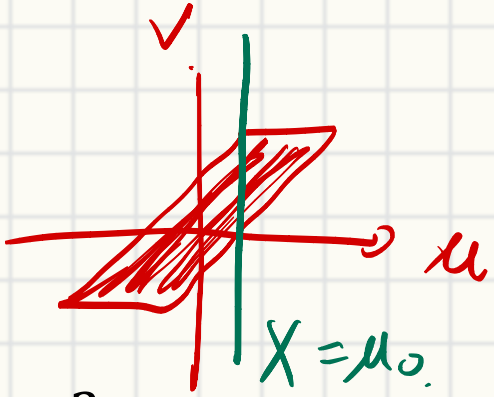
- Recap
  - Constant estimators
  - Unconstrained estimators
  - Linear estimators
- Examples

Joint Gaussian Distribution (Ch 4.11)

- Motivation
- Facts
- Examples

Estimators Recap

$E[(Y-c')^2] \geq Var(Y)$



- Constant estimator
  - $c^* = E[Y]$, MSE = $Var(Y)$
- Unconstraint estimator
  - $g^*(X) = E[Y|X]$ MSE=$E[Y^2] - E[(E[Y|X])^2]$
  - Best estimator, but requires $f_{Y|X}$

$E[(Y-g(X))^2] \geq E[(Y-g^*(X))^2]$ $\quad ||$

- Linear estimator
  - $\underbrace{\hat{E}[Y|X]}_{=aX+b} = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y(\frac{X-\mu_X}{\sigma_X})$
  - MSE= $\sigma_Y^2 - \frac{(Cov(X,Y))^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$

$$\hat{E}[Y|X] = aX + b$$

$$\underset{a,b}{\operatorname{argmin}} \; E[(Y - \hat{E})^2] = E[(\overbrace{Y - aX}^{Z} - b)^2]$$

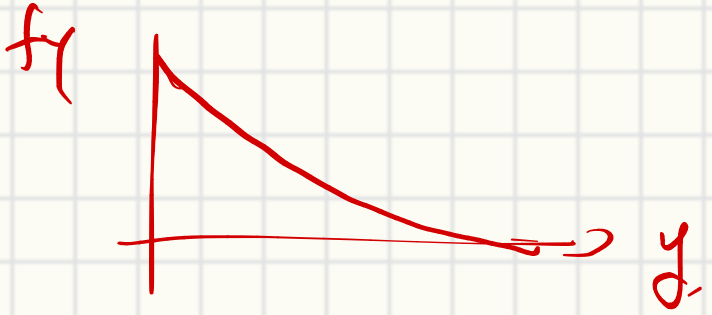
$$b^* = E[Z] = \mu_Y - a\mu_X$$

$$\Rightarrow E[(Y - aX - b)^2] = \underbrace{E[((Y - \mu_Y) - a(X - \mu_X))^2]}_{f}$$

$$\frac{df}{da} = 0 \quad \Rightarrow \quad a^* = \frac{Cov(X,Y)}{Var(X)}$$

Example

$Y = X + N$



Let $X = Y + N$, $Y \sim Exp(\lambda)$ and $N \sim N(0, \sigma_N^2)$. Assume Y and N are independent

- Find $\hat{E}[Y|X]$

$$\begin{cases} \mu_Y = \frac{1}{\lambda} \\ \mu_X = \mu_Y + \mu_N = \frac{1}{\lambda} + 0 = \frac{1}{\lambda} \\ Cov(X,Y) = Cov(Y + \underbrace{N, Y}_{0}) = Var(Y) = \frac{1}{\lambda^2} \\ Var(X) \end{cases}$$

$$= Cov(Y+N, Y+N) = Var(Y) + Var(N) = \frac{1}{\lambda^2} + \sigma_N^2$$

~~best~~

- Find the unconstrained estimator of Y

better

$$\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X)$$

$$= \frac{1}{\lambda} + \frac{1}{1+\lambda^2\sigma_N^2}\left(X - \frac{1}{\lambda}\right) = \frac{X + \lambda\sigma_N^2}{1+\lambda^2\sigma_N^2}$$

$Y > 0$, $\quad \hat{E}[Y|X] = aX+b \not> 0$,

$$g(X) = \max(0, \hat{E}[Y|X])$$

$$E[(Y-g(X))^2] < E[(Y-\hat{E}[Y|X])^2]$$

Example

Suppose X and Y are uniformly distributed in the triangle.

- Find $g^*(u) = E[Y|X = u]$ and the corresponding minimum MSE

$$g^*(u) = \begin{cases} 0.75+0.75u & -1<u\le 0 \\ 0.75+0.25u & 0<u<1 \\ \text{undefined} & \text{else} \end{cases}$$

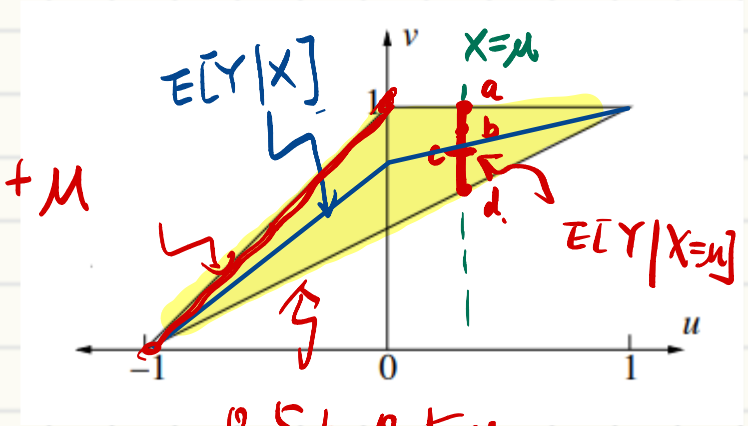
- Find $\hat{E}[Y|X = u]$, and compute MSE

$$E[Y|X=u] = \int v f_{Y|X}(v|u)\,dv$$

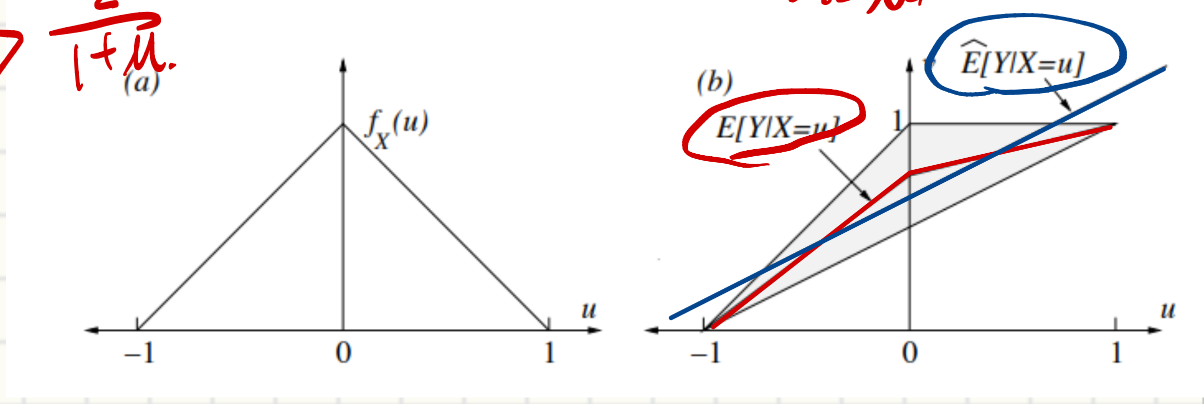
$\to \frac{2}{1+u}$

$\mu_X, \mu_Y, Cov(X,Y), Var(X)$

$1+u$



$0.5+0.5u$



Final Review

Next Wednesday class

↳ Input top 5 ~

↳ Final composition

60% 50% pre midterm 2

50% post midterm 2 (comprehensive)

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