

Last lecture

Joint pdfs of functions of RV (Ch 4.7)

- Linear mapping function (Ch 4.7.1)
- One to one/ Multiple to one functions (Ch 4.7.2-3)
 - Will not be tested

Correlation and covariance (Ch 4.8)

- Definition
- Properties
- Examples

Agenda

Correlation and covariance ([Ch 4.8](#))

- Examples

Minimum mean square error estimation ([Ch 4.9](#))

- Constant estimators
- Unconstrained estimators
- Linear estimators

Example

$$\begin{aligned} \text{Cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[XY] - \mu_X \mu_Y \end{aligned}$$

$$\rho_{X,Y} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Simplify the following expressions:

- $\text{Cov}(8X + 3, 5Y - 2)$
- $\text{Cov}(10X - 5, -3X + 15)$
- $\text{Cov}(X + 2, 10X - 3Y)$
- $\rho_{10X, Y+4}$

Example

Suppose the covariance matrix of RV vector $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ is $\begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

- Find $Cov(X_1 + X_2, X_1 + X_3)$
- Find a s.t. $X_2 - aX_1$ is uncorrelated with X_1
- Find ρ_{X_1, X_2}
- Find $Var(X_1 + X_2 + X_3)$

Minimum Mean Square Error Estimation

Constant Estimator

Given RV Y , if we know f_Y

- Constant estimator: constant $\delta^* = \operatorname{argmin}_{\delta} E[(Y - \delta)^2]$
- $E[(Y - \delta)^2] = E[Y^2] - 2\delta E[Y] + \delta^2$
- Taking derivative w.r.t δ
- $\delta^* =$
- $\text{MSE } E[(Y - \delta^*)^2] =$

Unconstrained Estimator

Given RV X, Y , if we know $f_{X,Y}$ and some observations X

- Unconstrained Estimator - $g^*(X) = \operatorname{argmin}_g E \left[(Y - g(X))^2 \right]$
- For example, say $X = 10$
 - $g^*(X) = E[Y|X = 10] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u = 10) dv$
 - $\text{MSE} = E[(Y - E[Y|X = u])^2 | X = u] =$
 - General form
 - $\text{MSE} E[(Y - E[Y|X])^2] =$

Unconstrained Estimator

What if we do not know $f_{X,Y}$ or $f_{Y|X}$ is hard to compute?

- Let $g^*(X) = aX + b$, find $\operatorname{argmin}_{(a,b)} E[(Y - (aX + b))^2]$
- Can also be written as $\operatorname{argmin}_{(a,b)} E[((Y - aX) - b)^2]$
 - b is the constant estimator of $Y - aX$
 - $b =$

Unconstrained Estimator

$$b = \mu_Y - a\mu_X$$

What if we do not know $f_{X,Y}$ or $f_{Y|X}$ is hard to compute?

- Let $g^*(X) = aX + b$, find $\operatorname{argmin}_{(a,b)} E[(Y - (aX + b))^2]$
- $\text{MSE } E[(Y - \mu_Y - a(X - \mu_X))^2] = \text{Var}(Y - aX) =$
- $\hat{E}[Y|X] = \mu_Y + \frac{\text{Cov}(X,Y)}{\text{Var}(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y\left(\frac{X - \mu_X}{\sigma_X}\right)$
- $\text{MSE} = \sigma_Y^2 - \frac{(\text{Cov}(X,Y))^2}{\text{Var}(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$

Example

Let $X = Y + N$, $Y \sim \text{Exp}(\lambda)$ and $N \sim N(0, \sigma_N^2)$. Assume Y and N are independent

- Find $\hat{E}[Y|X]$
- Find the unconstrained estimator of Y

Example

Suppose X and Y are uniformly distributed in the triangle.

- Find $g^*(u) = E[Y|X = u]$ and the corresponding minimum MSE
- Find $\hat{E}[Y|X = u]$, and compute MSE

