Last lecture

Joint pdfs of functions of RV (Ch 4.7)

$$f_{W,Z}(\alpha,\beta) = \frac{1}{\det(A)}$$

- Linear mapping function (Ch 4.7.1)
- One to one/ Multiple to one functions (Ch 4.7.2-3)
  - Will not be tested

$$f_{X,Y}(u,v)$$

Correlation and covariance (Ch 4.8)

- Definition
- Properties
- Examples

Agenda

Correlation and covariance (Ch 4.8)

- Examples

Minimum mean square error estimation (Ch 4.9)

- Constant estimators
- Unconstrained estimators
- Linear estimators

Example

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Simplify the following expressions:

- $Cov(8X + \cancel{3}, 5Y - \cancel{2})$ $= Cov(8X, 5Y) = 40\, Cov(X,Y)$
- $Cov(10X - \cancel{5}, -3X + \cancel{15})$ $= Cov(10X, -3X) = -30\, Var(X)$
- $Cov(X + \cancel{2}, 10X - 3Y)$ $= 10\, Var(X) - 3\, Cov(X,Y)$
- $\rho_{10X, Y+4}$ $= \dfrac{Cov(10X, Y+\cancel{4})}{\sigma_{10X}\, \sigma_{Y+\cancel{4}}} = \dfrac{\cancel{10}\, Cov(X,Y)}{\cancel{10}\, \sigma_X\, \sigma_Y}$

$= \rho_{X,Y}$

Remark

$\rho_{-10X, Y} = -\rho_{X,Y}$

Example

$c_{11} = Var(X_1)$

$c_{12} = Cov(X_1, X_2)$

$c_{ij} = Cov(X_i, X_j)$

Suppose the covariance matrix of RV vector $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ is $\begin{bmatrix} \boxed{5} & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

- Find $Cov(X_1 + X_2, X_1 + X_3)$

$= Var(X_1) + Cov(X_1, X_3) + Cov(X_2, X_1) + Cov(X_2, X_3)$

$= 5 + 0 + 2 + 2 = 9$

- Find a s.t. $X_2 - aX_1$ is uncorrelated with X_1

$Cov(X_2 - aX_1, X_1) = 0$

$Cov(X_2, X_1) - a\,Var(X_1) = 0$

$2 - a \times 5 = 0 \qquad a = \frac{2}{5}$

- Find ρ_{X_1, X_2}

- Find $Var(X_1 + X_2 + X_3)$

$Cov(X_1 + X_2 + X_3, X_1 + X_2 + X_3) = 5 + 2 + 0 + 2 + 5 + 2 + 0 + 2 + 5$
$= 23$

$$\rho_{X_1,X_2} = \frac{\operatorname{cov}(X_1, X_2)}{\sigma_{X_1}\underbrace{\sigma_{X_2}}} = \frac{2}{\sqrt{5}\,\sqrt{5}} = \frac{2}{5}$$

Minimum Mean Square Error Estimation

Constant Estimator

Minimum Mean Square Error

Given RV Y, if we know f_Y

- Constant estimator: constant $\delta^* = argmin_\delta E[(Y-\delta)^2]$
- $E[(Y-\delta)^2] = E[Y^2] - 2\delta E[Y] + \delta^2$
- Taking derivative w.r.t δ find extrema. $\Rightarrow 2\delta - 2E[Y] = 0$

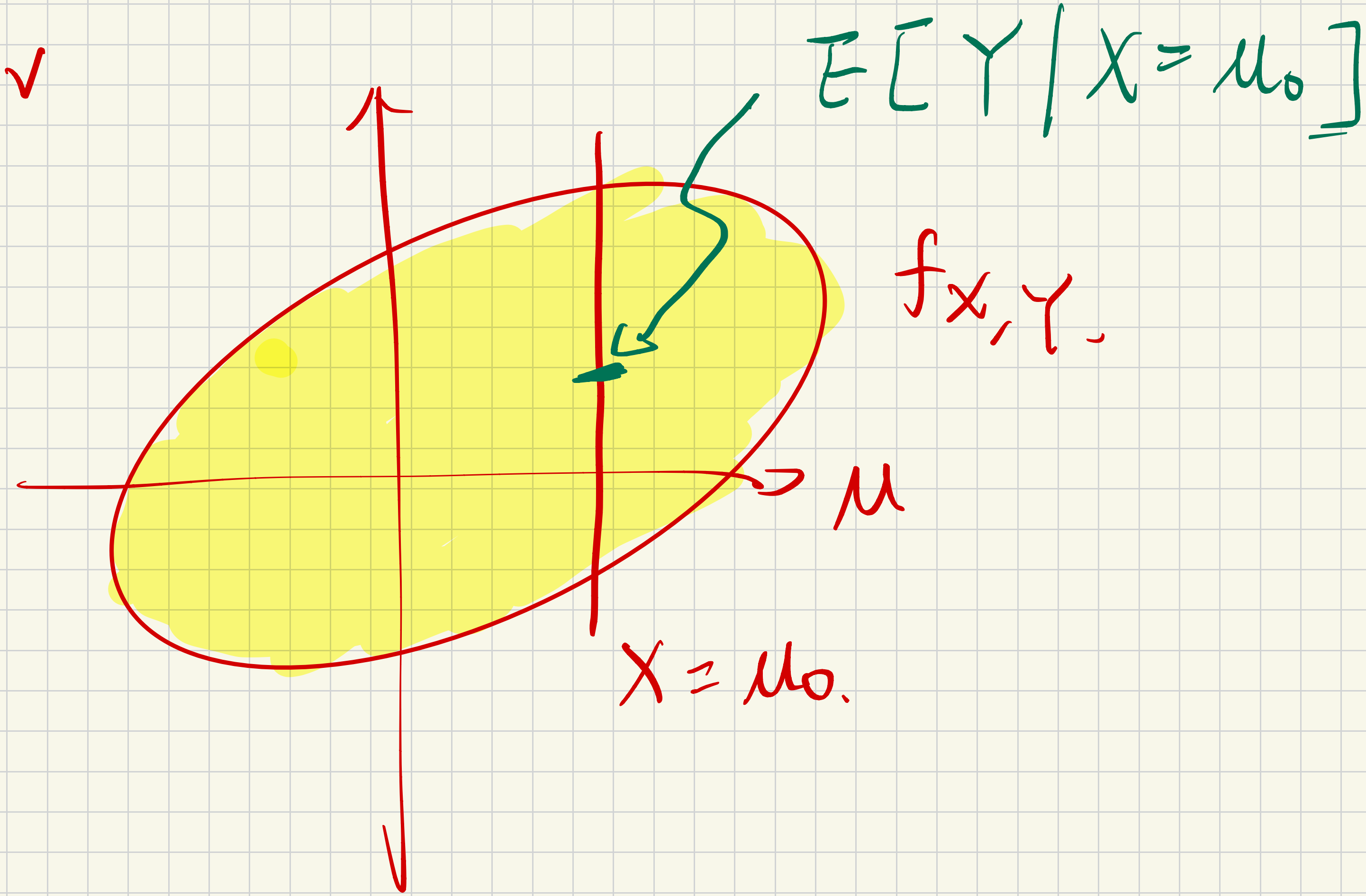
  $\delta^* = E[Y] = \mu_Y$
- $\delta^* = \mu_Y$
- MSE $E[(Y-\delta^*)^2] = E[(Y-\mu_Y)^2] = Var(Y)$

Unconstrained Estimator

$g(X)$ as MMSE estimator of Y.

Given RV X, Y, if we know $f_{X,Y}$ and some observations X

- Unconstrained Estimator - $g^*(X) = argmin_g \, E\left[\left(Y - g(X)\right)^2\right]$
- For example, say $X = 10$
  - $g^*(X) = E[Y|X = 10] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u = 10)dv$
  - MSE = $E[(Y - \underbrace{E[Y|X = u]}_{g^*(X)})^2|X = u] =$ $E[Y^2|X=u] - (E[Y|X=u])^2$
  - General form $g^*(X) = E[Y|X]$
  - MSE $E[(Y - E[Y|X])^2] =$ $E[Y^2|X] - E[(\underbrace{E[Y|X]}_{R.V.})^2]$


$E[Y|X=u_0]$
$f_{X,Y}$
u
$X = u_0$

~~Unconstrained~~ Estimator

Linear

$f_{Y|X}$ is not available

hard to compute ?

What if we do not know $f_{X,Y}$ or $f_{Y|X}$ is hard to compute?

- Let $g^*(X) = aX + b$, find $argmin_{(a,b)}\, E[(Y - (aX + b))^2]$
- Can also be written as $\boxed{argmin_{(a,b)}\, E[(\underbrace{(Y - aX)}_{RV} - \underline{b})^2]}$
  - b is the constant estimator of $Y - aX$
  - $b^* = E[Y - aX]$

$= \underline{\mu_Y - a\mu_X}$

constant estimator for

$Y - aX$

~~Unconstrained~~ Estimator

$b = \mu_Y - a\mu_X$

Linear

What if we do not know $f_{X,Y}$ or $f_{Y|X}$ is hard to compute?

- Let $g^*(X) = aX + b$, find $argmin_{(a,b)}\, E[(Y - (aX + b))^2]$
- MSE $E\left[\left(Y - \mu_Y - a(X - \mu_X)\right)^2\right] = Var(Y - aX) =$

b

$Cov(Y - aX, Y - aX)$

Take derivative w.r.t. a

$Var(Y) - 2a\,Cov(X, Y) + a^2\,Var(X)$

$2a\,Var(X) - 2\,Cov(X, Y) = 0$

- $\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y\left(\frac{X - \mu_X}{\sigma_X}\right)$

$\boxed{a^* = \frac{Cov(X, Y)}{Var(X)}}$

Standardized X

- MSE = $\sigma_Y^2 - \frac{(Cov(X,Y))^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$