Last lecture

More examples on joint RVs (Ch 4.6)

- Buffon's needle problems
- Maximum likelihood estimator

Joint pdfs of functions of RV (Ch 4.7)

- Linear mapping function (Ch 4.7.1)

Agenda

Joint pdfs of functions of RV (Ch 4.7)

- Linear mapping function (Ch 4.7.1)
- One to one/ Multiple to one functions (Ch 4.7.2-3)
    - Will not be tested

Correlation and covariance (Ch 4.8)

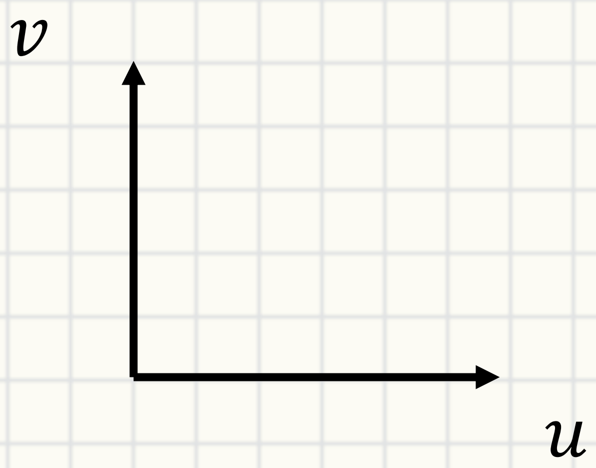
- Definition
- Properties
- Examples

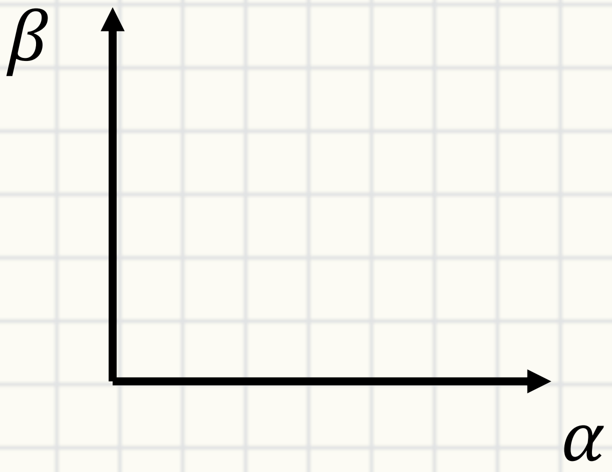
Joint PDF properties

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $\det(A) \neq 0$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v)$

- Intuition - $W = 2X + Y, Z = X - Y$





Example

$W = X - Y, Z = X + Y$. Express $f_{W,Z}(\alpha, \beta)$ in terms of $f_{X,Y}$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$

- $Det(A) =$

- For $(W, Z) = (\alpha, \beta)$, $(X, Y) =$

- $f_{W,Z}(\alpha, \beta) =$

Example

Suppose X and Y are continuous independent RVs.

- $W = X + Y, Z = Y$
- Find$f_{W,Z}(\alpha, \beta)$ and $f_W(\alpha)$

Correlation and covariance

Definition

Metrics such as mean/ variance is more convenient than PDF/ CDF

- Recall $\mu_X = E[X], Var(X) = E[(X - \mu_X)^2]$
- For jointly distributed X and Y
  - Covariance $Cov(X, Y) =$
  - Correlation coefficient $\rho_{X,Y} =$
    - $\rho_{X,Y} \in$
  - Cross moment $E[XY]$ (less used)

Definition

- $Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$
  $=$

- **Uncorrelated** -
- $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$
  - $\uparrow$: X and Y has the same trend
  - $\rho_{X,Y} > 0$: Positively correlated

Properties

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Some properties for independent and uncorrelated

- Independent uncorrelated
  - $E[XY] =$
- Uncorrelated independent
- Multiple RVs are uncorrelated if they are pairwise uncorrelated

$$Cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$$
$$= E[XY] - \mu_X \mu_Y$$

Properties

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Some properties for independent and uncorrelated

- $Cov(X + Y, U + V) = Cov(X,U) + Cov(X,V) + Cov(Y,U) + Cov(Y,V)$

- $Cov(aX + b, cY + d) = acCov(X,Y)$

- If X and Y are independent
  - $Var(X + Y) = Cov(X + Y, X + Y) = Cov(X,X) + Cov(Y,Y)$

Example

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Simplify the following expressions:

- $Cov(8X+3,5Y-2)$
- $Cov(10X-5,-3X+15)$
- $Cov(X+2,10X-3Y)$
- $\rho_{10X,Y+4}$

Example

Suppose the covariance matrix of RV vector $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$ is $\begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 2 \\ 0 & 2 & 5 \end{bmatrix}$

- Find $Cov(X_1 + X_2, X_1 + X_3)$
- Find a s.t. $X_2 - aX_1$ is uncorrelated with X_1
- Find ρ_{X_1,X_2}
- Find $Var(X_1 + X_2 + X_3)$