Last lecture

More examples on joint RVs (Ch 4.6)

- Buffon's needle problems
- Maximum likelihood estimator

Joint pdfs of functions of RV (Ch 4.7)

- Linear mapping function (Ch 4.7.1)

Agenda

Joint pdfs of functions of RV (Ch 4.7)

- Linear mapping function (Ch 4.7.1)
- One to one/ Multiple to one functions (Ch 4.7.2-3)
  - Will not be tested

Correlation and covariance (Ch 4.8)

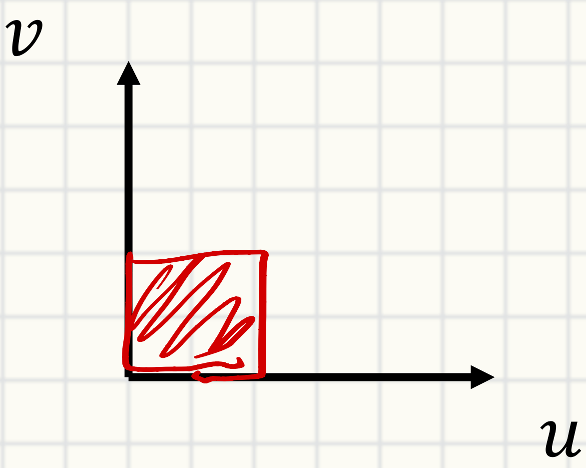
- Definition
- Properties
- Examples

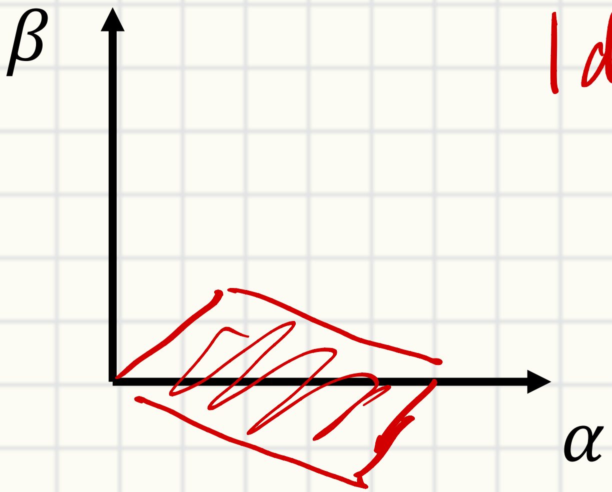
Joint PDF properties

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $\det(A) \neq 0$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v)$
- Intuition - $W = 2X + Y, Z = X - Y$

$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$





$|\det(A)|$
$= |-2-1|$
$= 3.$

Example

$W = X - Y, Z = X + Y$. Express $f_{W,Z}(\alpha,\beta)$ in terms of $f_{X,Y}$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$
- $Det(A) =$ $1 \times 1 - (-1) \times 1 = 2$
- For $(W,Z) = (\alpha,\beta)$, $(X,Y) =$ $\left(\frac{\alpha+\beta}{2}, \frac{-\alpha+\beta}{2}\right)$
- $f_{W,Z}(\alpha,\beta) =$

$$\frac{1}{|2|} \times f_{XY}\left(\frac{\alpha+\beta}{2}, \frac{-\alpha+\beta}{2}\right)$$

$$\begin{aligned} W &= X - Y \\ +)\ Z &= X + Y \\ \hline W + Z &= 2X \end{aligned}$$

$X = \frac{W+Z}{2}$

$Y = \frac{-W+Z}{2}$

Example

Suppose X and Y are continuous independent RVs.

- $W = X + Y, Z = Y$
- Find $f_{W,Z}(\alpha, \beta)$ and $f_W(\alpha)$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\det(A) = 1 \times 1 - 0 \times 1 = 1$

$(W, Z) = (\alpha, \beta) \rightarrow (X, Y) = (\alpha - \beta, \beta)$

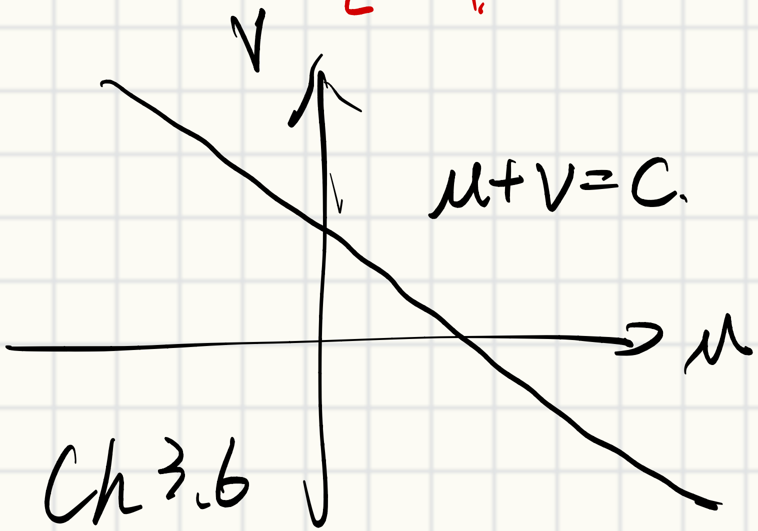
$\begin{cases} W = X + Y \\ Z = Y \end{cases}$

$f_{W,Z}(\alpha, \beta) = \frac{1}{1} \times f_{XY}(\alpha - \beta, \beta)$

$f_W(\alpha) = \int_{-\infty}^{\infty} f_{XY}(\alpha - \beta, \beta)\, d\beta$

$=$

$f_S(c)$ in Ch 3.6



Correlation and covariance

Definition

Metrics such as mean/ variance is more convenient than PDF/ CDF

- Recall $\mu_X = E[X], Var(X) = E[(X - \mu_X)^2]$
- For jointly distributed X and Y
  - Covariance $Cov(X, Y) = E[(X-\mu_X)(Y-\mu_Y)]$
  - Correlation coefficient $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$
    - $\rho_{X,Y} \in [-1, 1]$

Not emphasized ~~•~~ Cross moment $E[XY]$ (less used)

(cross correlation in signal processing)

Definition

$E[XY - \mu_X Y - X\mu_Y + \mu_X \mu_Y]$

same

constant

- $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

  $= E[X(Y-\mu_Y)] = E[(X-\mu_X)Y]$

  $= E[XY] - \mu_X \mu_Y$

- **Uncorrelated** - $Cov(X,Y) = 0$.
- $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$
  - $\uparrow$: X and Y has the same trend
  - $\rho_{X,Y} > 0$: Positively correlated

    < 0 : Negatively corr.

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y$$

Properties

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Some properties for independent and uncorrelated

- Independent implies uncorrelated
  - $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} uv f_{XY}(u,v)\,du\,dv = \int_{-\infty}^{\infty} u f_X(u)\,du \int_{-\infty}^{\infty} v f_Y(v)\,dv = E[X]E[Y]$
- Uncorrelated does not imply independent
- Multiple RVs are uncorrelated if they are pairwise uncorrelated

pairwise independent weaker than independent

pairwise uncorrelated = uncorrelated

Properties

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[XY] - \mu_X\mu_Y$$

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$

Some properties for independent and uncorrelated

- $Cov(X+Y, U+V) = Cov(X,U) + Cov(X,V) + Cov(Y,U) + Cov(Y,V)$
- $Cov(aX+b, cY+d) = acCov(X,Y)$

  $\rho_{aX+b,cY+d} = \rho_{X,Y}$
- If X and Y are independent
  - $Var(X+Y) = Cov(X+Y, X+Y) = Cov(X,X) + Cov(Y,Y)$

    $= Var(X) + Var(Y)$

if $\forall x$ $E[Y|x]=0$ $\rightarrow$ $E[X(Y-\mu_Y)]=0$

$E[XY]-\mu_X\mu_Y=0$

Slido

Select those are uncorrelated

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