Last lecture

Sum of joint RVs (Ch 4.5)

- Continuous RVs & Examples
- Sum of Gaussians

More examples on joint RVs (Ch 4.6)

- Max of two RVs
- Buffon's needle problems
- Maximum likelihood estimator

Agenda

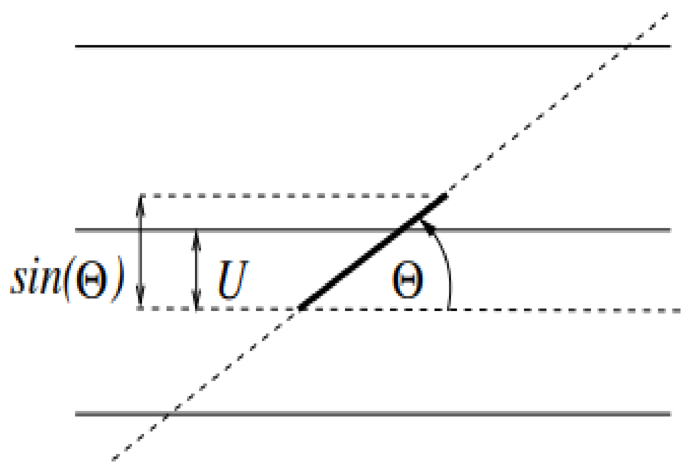
More examples on joint RVs (Ch 4.6)

- Buffon's needle problems
- Maximum likelihood estimator

Joint pdfs of functions of RV (Ch 4.7)

- Linear mapping function (Ch 4.7.1)
- One to one/ Multiple to one functions (Ch 4.7.2-3)
  - Will not be tested

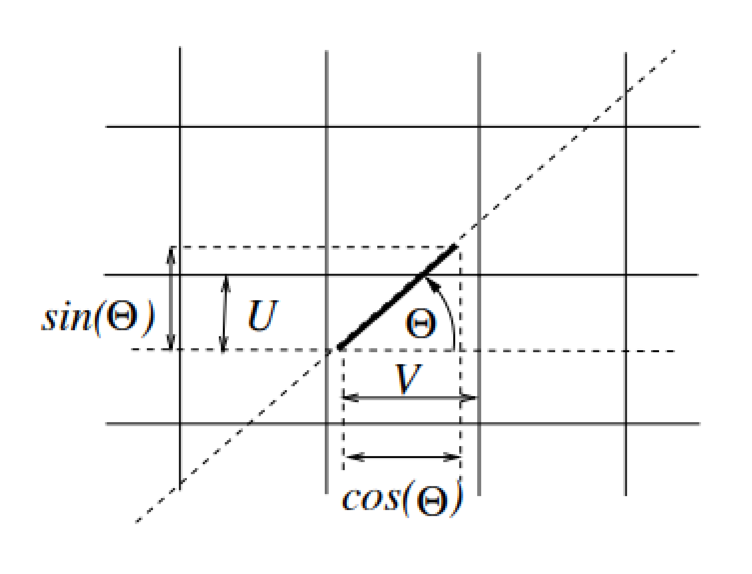
Buffon's needle problem



- Draw many parallel horizontal lines
  - Space 1 inch between two lines
  - Throw a needle of 1 inch length on the plane
  - Find P\{ "The needle intersect with a line" \}

Define U = "distance from the needle lower end to the first line above

Buffon’s needle problem (2)



- What if there are “horizontal” and “vertical” lines?

Let M_h denotes “missing horizontal lines”
M_v denotes “missing vertical lines”

Maximum Likelihood Estimator

A drone is accelerating constantly with unknown rate

- At time t, the location is bt
- Measurement $X_t = bt + W_t$
    - $W_t \sim N(0,1)$ is the independent random noise
- Given $X_{1:T} = u_{1:T}$ as the observation, find $\hat{b}_{ML}$
- Is $\hat{b}_{ML}$ **unbiased**, i.e., $\hat{b}_{ML}(\mu_{1:T}) = b$

Joint pdfs of functions of RV

Notation and Definition

Denote the point on the plane (X, Y) as a column vector $\begin{pmatrix} X \\ Y \end{pmatrix}$

- $f_{X,Y}(u, v)$ is denoted as $f_{X,Y}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$

Suppose $W = aX + bY$ and $Z = cX + dY$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- For any $\begin{pmatrix} X \\ Y \end{pmatrix}$ in $u - v$ plane, we can find $\begin{pmatrix} W \\ Z \end{pmatrix}$ in $\alpha - \beta$ plane

Determinant and Inverse

$\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$

- $\det(A) = ad - cb.$ If $\det(A) \neq 0$
  - $\alpha - \beta$ span a plane

  - $\begin{pmatrix} u \\ v \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, where $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

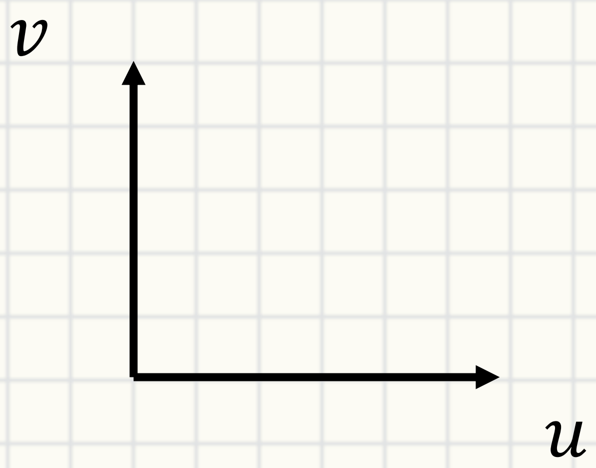
  - $|\det(A)|$ is like “ “

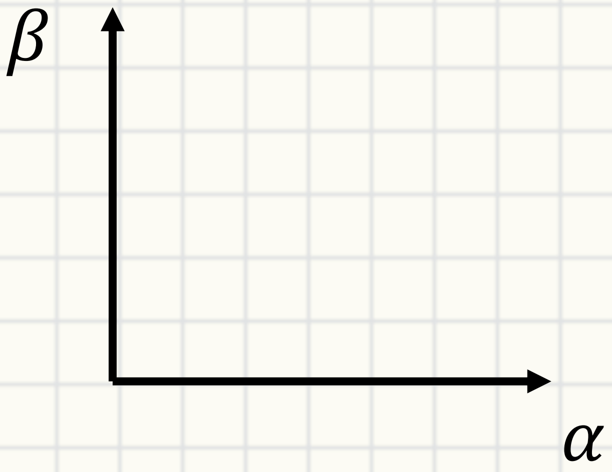
Joint PDF properties

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $\det(A) \neq 0$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v)$

- Intuition - $W = 2X + Y, Z = X - Y$





Example

$W = X - Y, Z = X + Y$. Express $f_{W,Z}(\alpha, \beta)$ in terms of $f_{X,Y}$

- $\binom{W}{Z} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \binom{X}{Y}$

- $Det(A) =$

- For $(W, Z) = (\alpha, \beta)$, $(X, Y) =$

- $f_{W,Z}(\alpha, \beta) =$

Example

Suppose X and Y are continuous independent RVs.

- $W = X + Y, Z = Y$
- Find$f_{W,Z}(\alpha, \beta)$ and $f_W(\alpha)$