Last lecture

Sum of joint RVs (Ch 4.5)

- Continuous RVs & Examples
- Sum of Gaussians

More examples on joint RVs (Ch 4.6)

- Max of two RVs
- Buffon’s needle problems
- Maximum likelihood estimator

Agenda

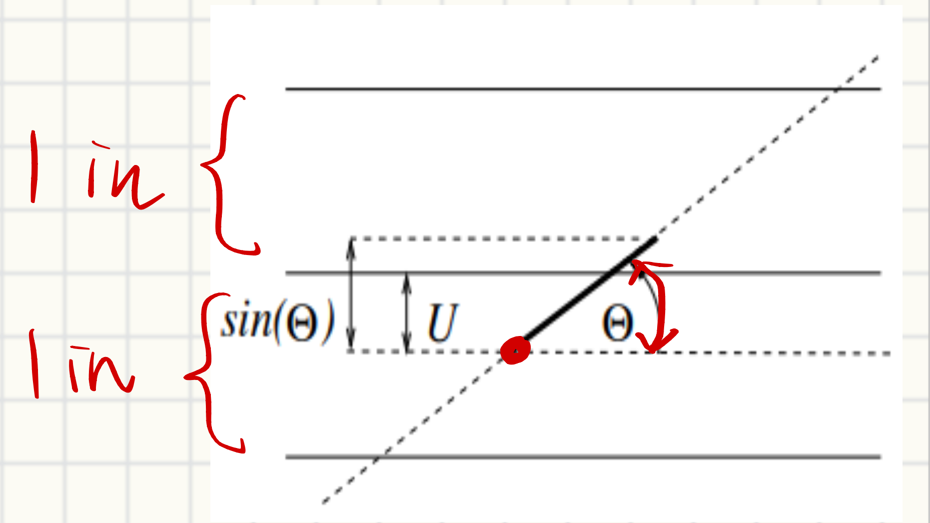
More examples on joint RVs (Ch 4.6)

- Buffon's needle problems
- Maximum likelihood estimator

Joint pdfs of functions of RV (Ch 4.7)

- Linear mapping function (Ch 4.7.1)
- One to one/ Multiple to one functions (Ch 4.7.2-3)
  - Will not be tested

Buffon's needle problem



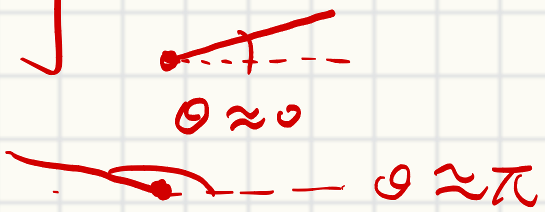
- Draw many parallel horizontal lines
  - Space 1 inch between two lines
  - Throw a needle of 1 inch length on the plane
  - Find P{ "The needle intersect with a line" }

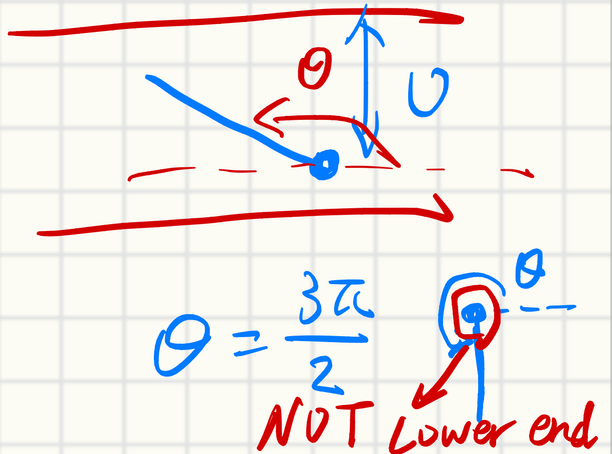
I

Define U = "distance from the needle lower end to the first line above

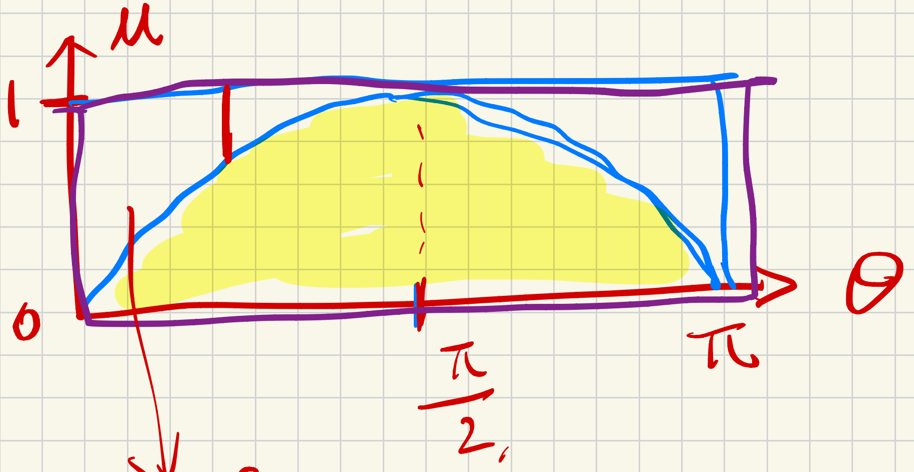
$U \sim$ Uniform [0, 1 (inch)]

$\theta \sim$ Uniform [0, π]





$$P\{I\} = P\{U \le \sin\theta\}$$



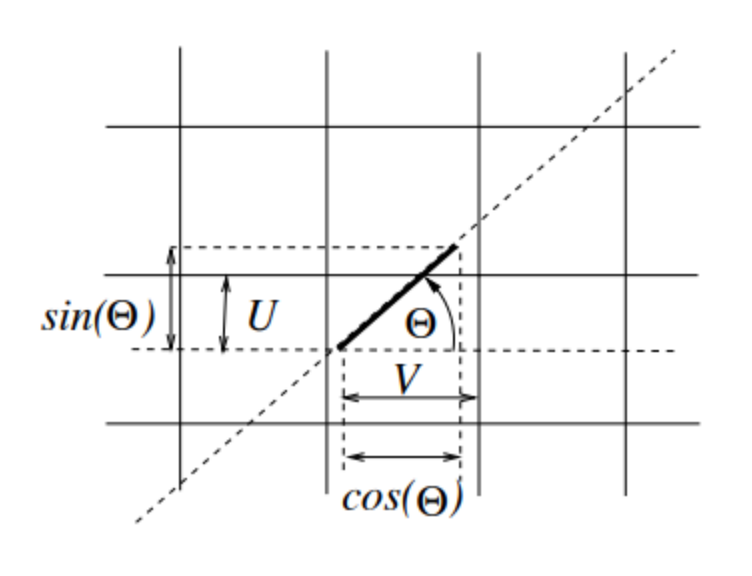
$$= \int_0^{\pi}\int_0^{\sin\theta} \overset{\frac{1}{\pi}}{f_{U,\Theta}(u,\theta)}\, du\, d\theta$$

$$f_{U,\Theta}(u,\theta) = \frac{1}{|S|} = \frac{1}{\pi}$$

$$= \frac{2}{\pi} \approx 0.63$$

Buffon's needle problem (2)

- What if there are "horizontal" and "vertical" lines?



Let M_h denotes "missing horizontal lines"
M_v denotes "missing vertical lines"

Wrong guess — $P\{$"Intersect vertical"$\}$ and $P\{$"Intersect horizontal"$\}$ is independent

$$P\{\text{"No Intersect"}\} = \left(1-\frac{2}{\pi}\right)\left(1-\frac{2}{\pi}\right) \approx 0.132$$

$$P(M_v \cap M_h) = \frac{1}{\pi}\int_0^{\pi} (1-\sin\theta)(1-|\cos\theta|)\,d\theta$$

$$= 1 - \frac{3}{\pi} \approx 0.045$$

Maximum Likelihood Estimator

accending

A drone is ~~accelerating~~ constantly with unknown rate

- At time t, the location is bt
- Measurement $X_t = bt + W_t$
    - $W_t \sim N(0,1)$ is the independent random noise
- Given $X_{1:T} = u_{1:T}$ as the observation, find $\hat{b}_{ML}$
- Is $\hat{b}_{ML}$ **unbiased**, i.e., $\hat{b}_{ML}(\mu_{1:T}) = b$

$$f_{X_t} \sim N(0,1) + bt$$

$$X_{1:t} \triangleq [X_1, X_2 \dots X_t]$$

$$\hat{b}_{ML} = \frac{\sum u_i t_i}{\sum t_i^2}$$

Maximum Likelihood Estimator

A drone is accelerating constantly with unknown rate

- $X_t \sim N(0,1) + bt = N(bt, 1)$
- $P(X_t = u_t | b) = f_{X_t}(u_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(u_t - bt)^2}{2}}$
- $f_{X_1, X_2, \dots, X_T}(u_1, \dots u_T) = \frac{1}{(2\pi)^{T/2}} e^{-\sum_{t=1}^{T} \frac{(u_t - bt)^2}{2}}$
- $\frac{d \log f_{X_1, \dots, X_T}}{db} = \sum_{t=1}^{T} (u_t - bt)\, t = 0$
- $\hat{b}_{ML} = \frac{\sum_{t=1}^{T} u_t t}{\sum_{t=1}^{T} t^2}$

$\mu_i = \underline{\mu_i} = \underline{bt}$ (because $E[W_t] = 0$.

$$\frac{\sum b t_i^2}{\sum t_i^2} = b$$

Joint pdfs of functions of RV

Notation and Definition

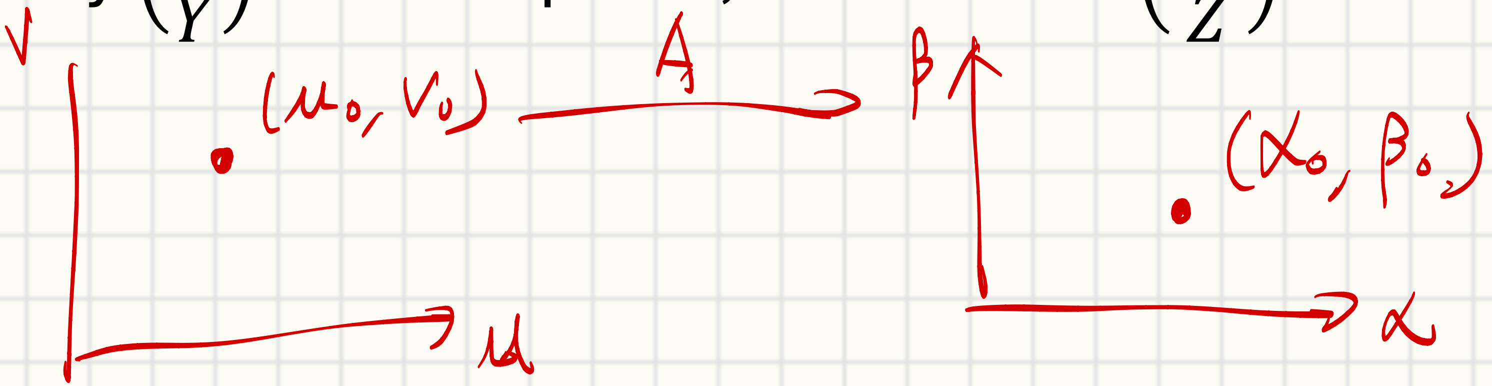
Denote the point on the plane (X, Y) as a column vector $\begin{pmatrix} X \\ Y \end{pmatrix}$

- $f_{X,Y}(u, v)$ is denoted as $f_{X,Y}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right)$

$$\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

Suppose $W = aX + bY$ and $Z = cX + dY$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- For any $\begin{pmatrix} X \\ Y \end{pmatrix}$ in $u - v$ plane, we can find $\begin{pmatrix} W \\ Z \end{pmatrix}$ in $\alpha - \beta$ plane



Determinant and Inverse

$\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$

- $\det(A) = ad - cb$. If $\det(A) \neq 0$
  - $\alpha - \beta$ span a plane

  - $\begin{pmatrix} u \\ v \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, where $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

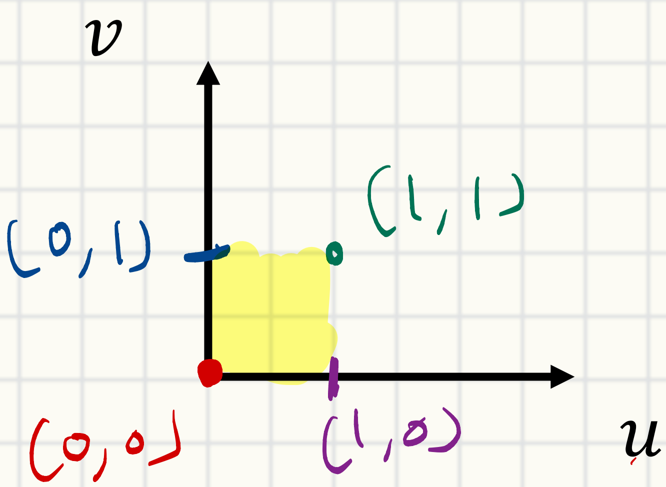
  - $|\det(A)|$ is like " scaling factor " (in single RV case)

Joint PDF properties

Suppose $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $\det(A) \neq 0$

- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v)$ $\quad A^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

- Intuition - $W = 2X + Y, Z = X - Y$ $\quad A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$



Area increase by $|\det(A)| = 3$