Last lecture

Independent RV (Ch 4.4)

- Examples

Sum of joint RVs (Ch 4.5)

- Motivation
- Discrete RVs & Examples

Agenda

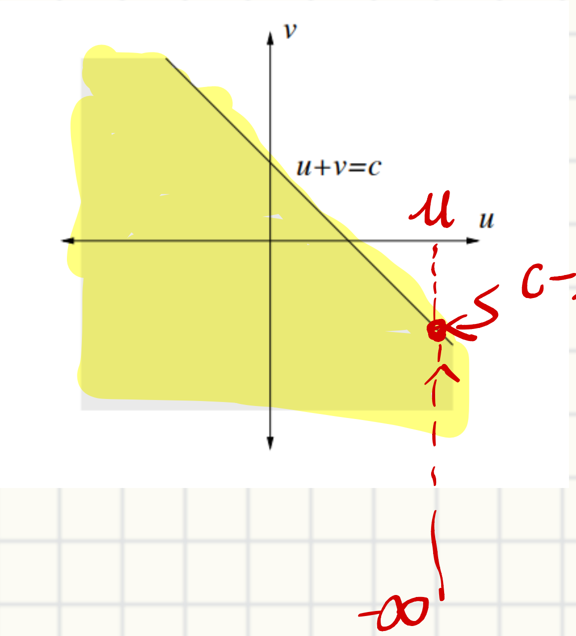
Sum of joint RVs (Ch 4.5)

- Continuous RVs & Examples
- Sum of Gaussians

More examples on joint RVs (Ch 4.6)

- Max of two RVs
- Buffon’s needle problems
- Maximum likelihood estimator

Sums of Continuous RVs



Let $S = X + Y$

- $F_S(c) = P\{S \le c\} = \int_{-\infty}^{\infty}\int_{-\infty}^{c-u} f_{XY}(u,v)\,dv\,du$
- $f_S(c) = \frac{dF_S(c)}{dc} = \int_{-\infty}^{\infty} \frac{d}{dc}\int_{-\infty}^{c-u} f_{XY}(u,v)\,dv\,du$

↳ $f_{XY}(u,c-u) - \underbrace{\cancel{f_{XY}(u,-\infty)}}_{\to 0}$

$= \int_{-\infty}^{\infty} f_{XY}(u,c-u)\,du$

⇒ "Integrate along the line"

If X and Y are independent

- $f_S(c) = \int_{-\infty}^{\infty} f_X(u)\, f_Y(c-u)\,du$

$= \underline{f_X * f_Y}$

Examples

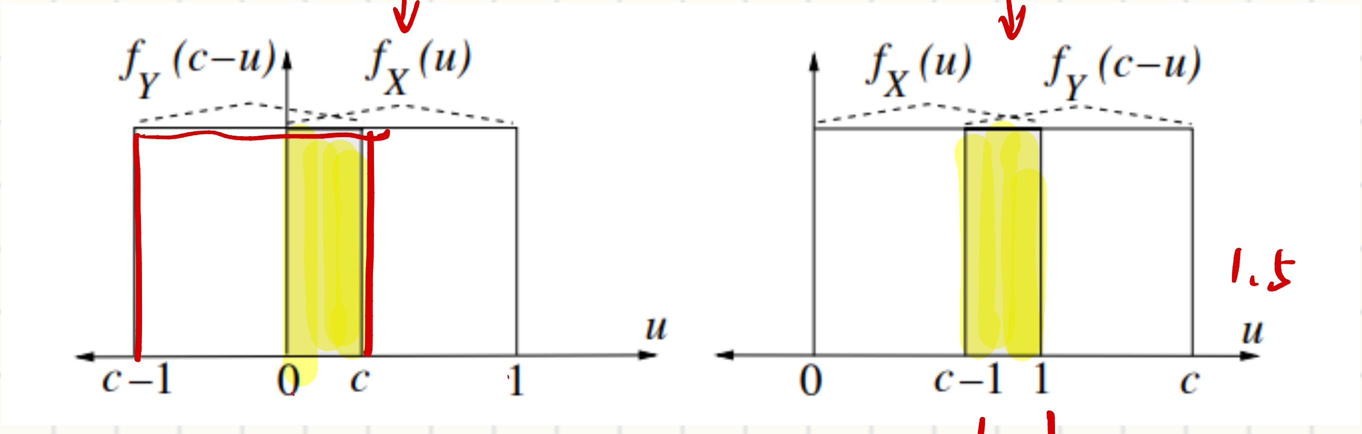
Suppose X and Y are independent, $X, Y \sim Uniform[0, 1]$. Find the pdf of $S = X + Y$

$\in [0, 2]$ $\quad 0 < c \leq 1 \quad\quad 1 < c \leq 2$

- $f_S = f_X * f_Y$
- What is $f_Y(c - u)$?

$\|$ $\quad 0 \to 1$

Uniform$[c-1, c]$

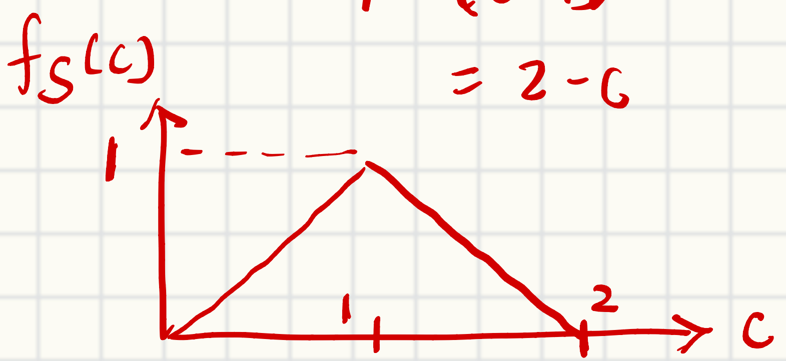


$1-(c-1)$

$= 2-c$

- If $0 < c \leq 1$, $f_X * f_Y =$ c
- If $1 < c \leq 2$, $f_X * f_Y =$ $2-c$

$$f_S(c) = \begin{cases} c & \text{if } c \in (0, 1] \\ 2-c & \text{if } c \in (1, 2] \\ 0 & \text{else} \end{cases}$$



Notes on Gaussian

$\boxed{\text{Exam Safe}}$

Diff. Mean

$\Rightarrow$ Gaussian Mixture

Assume $X \sim N(0, \sigma_1^2)$, $Y \sim N(0, \sigma_1^2)$

- Sum of two Gaussian of same mean
  - Mean keeps the same
  - $\sigma_S^2 = \sigma_X^2 + \sigma_Y^2$

  - Tedious proof in textbook formula (4.20)
  - But high-level idea - approximate by Binomials…

$$Var(B_x) = np(1-p)$$

$$Var(B_s) = (n_x + n_y)\, p(1-p) = Var(B_x) + Var(B_Y)$$

More examples on joint RVs

Max of two RVs

Let $W = \max(X, Y)$, X, Y are independent

- $F_W(t) = P\{W \le t\} = F_{XY}(t, t) = F_X(t) F_Y(t)$

  $\underbrace{W \le t}_{\text{Both } X, Y \le t}$

- $f_W(t) = \frac{dF_W(t)}{dt} = \frac{dF_X(t)}{dt} F_Y(t) + F_X(t) \frac{dF_Y(t)}{dt} = f_X(t) F_Y(t) + F_X(t) f_Y(t)$

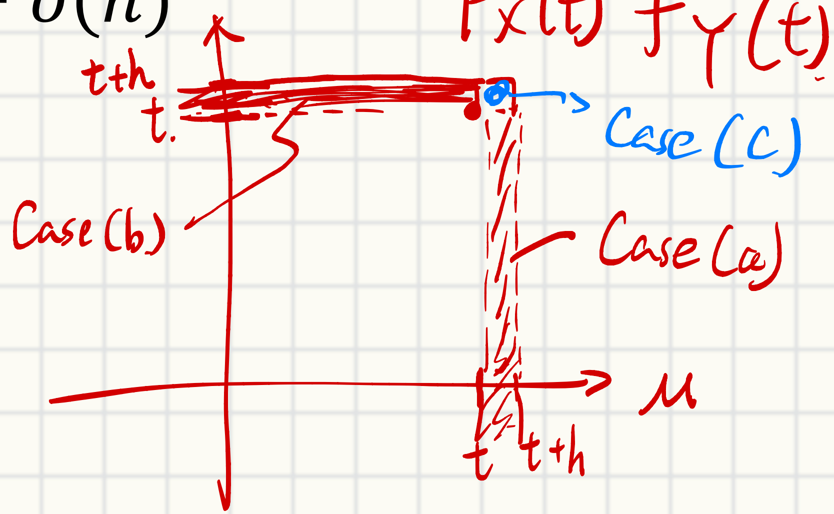
Abstract – on $P\{W \in (t, t+h]\} = f_W(t)h + o(h)$

- Case (a): $Y \le t, X \in (t, t+h]$
- Case (b): $X \le t, Y \in (t, t+h]$
- Case (c): $X \in (t, t+h], Y \in (t, t+h]$

$F_Y(t) f_X(t)$

$F_X(t) f_Y(t)$

$f_X(t) f_Y(t)$



Slido

$F_W^c(c) = P\{X > c, Y > c\}$
$= [1 - F_X(c)][1 - F_Y(c)]$

Let $W = \min(X, Y)$, X and Y are continuous independent RVs. What is $f_W(c)$?

$F_W(c) = 1 - [1 - F_X(c)][1 - F_Y(c)]$

(a) $f_X(c) + f_Y(c)$

(b) $f_X(c)F_Y(c) + f_Y(c)F_X(c)$

(c) $(1 - f_X(c))F_Y(c) + (1 - f_Y(c))F_X(c)$

$f_W = f_X(c)[1 - F_Y(c)] + f_Y(c)[1 - F_X(c)]$

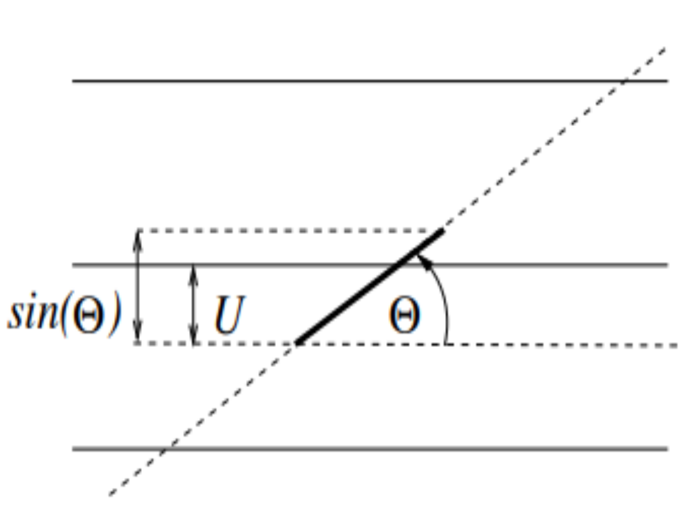
✓ (d) $\underbrace{f_X(c)}_{X \approx c}\underbrace{(1 - F_Y(c))}_{Y \geq c} + \underbrace{f_Y(c)}_{Y \approx c}\underbrace{(1 - F_X(c))}_{X \geq c}$

(e) $(1 - F_Y(c))(1 - F_X(c))$

(f) $(1 - f_X(c))(1 - F_Y(c)) + (1 - f_Y(c))(1 - F_X(c))$

#1892486

Buffon's needle problem



- Draw many parallel horizontal lines
  - Space 1 inch between two lines
  - Throw a needle of 1 inch length on the plane
  - Find P{ "The needle intersect with a line" } $= \frac{2}{\pi}$

Define U = "distance from the needle lower end to the first line above