Last lecture

Joint PDF (Ch 4.3)

- Example
  - Uniform distribution
  - Conditional distribution

Independent RV (Ch 4.4)

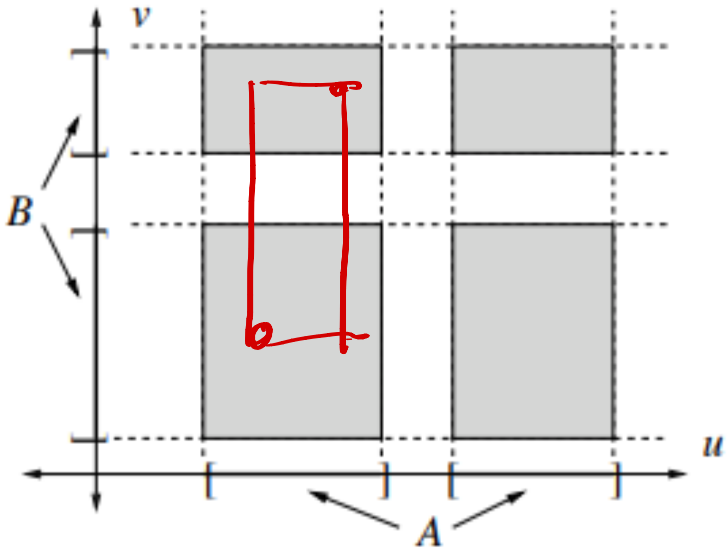
- From event to RV - CDF
- Check using PDF

Agenda

Independent RV (Ch 4.4)

- Examples

Product Set



Let A, B denote a finite union of intervals

- $|A|$ denotes the total length of $|A|$

The product set $A \times B = \{(u, v): u \in A, v \in B\}$

- The total area $|A \times B| = |A| \times |B|$

Swap property: $S \in \mathbb{R}^2$ has the swap property if

- For any pair of points $(a, b), (c, d) \in S$, (a, d) and (c, b) also in S

Proposition - $S \in \mathbb{R}^2$, S is a product set if and only if it has the swap property

Properties of independent

- If X, Y are independent and jointly continuous type RVs, then support of $f_{X,Y}$ is a product set

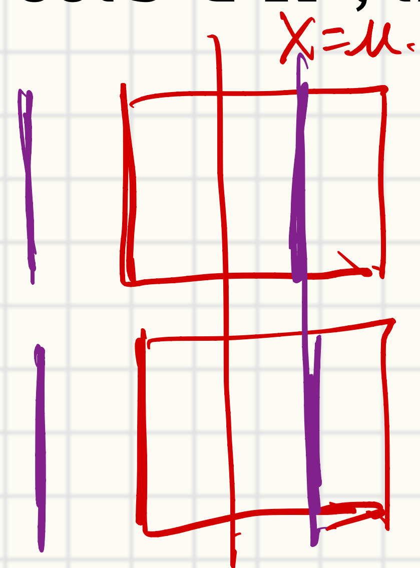
$$f_{XY} = f_X \times f_Y.$$

$$\hookrightarrow |A| \times |B| = |S|.$$

- Support X, Y are uniformly distributed over set $S \in \mathbb{R}^2$, then X and Y are independent iif S is a product set

if. $f_{Y|X} = f_Y$

only if.



Examples

Formal

$f_{Y|X} = \frac{f_{XY}}{f_X}$ $\qquad f_Y = \int_0^{1-v} f_{XY}(u,v)\,du$

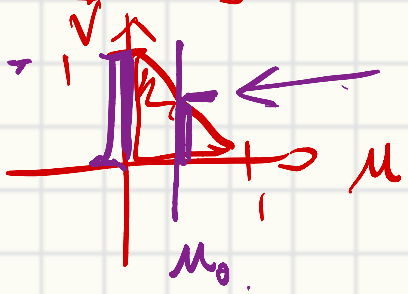
$f_X = \int_0^{1-u} f_{XY}(u,v)\,dv.$

Decide whether the if X and Y are independent if

- ✗ $f_{X,Y}(u,v) = Cu^2v^2$ for $u, v > 0$ and $\boxed{u + v \leq 1}$; 0 else

S

$f_{Y|X} = f_Y$? No.

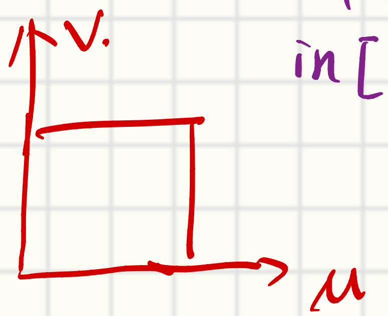


$f_Y \neq 0$ in $[0,1]$

$f_{Y|X} \neq 0$ in $[0, 1-u]$

- ✗ $f_{X,Y}(u,v) = u + v$ for $u, v \in [0,1]$; 0 else

✗ $u+v \neq g(u)\,g'(v)$



- ✓ $f_{X,Y}(u,v) = 9u^2v^2$ for $u, v \in [0,1]$; 0 else

$f_X(u) = \int_0^1 9u^2v^2\,dv = 3u^2v^3\Big|_0^1 = 3u^2$

$f_Y(u) = 3v^2 \Rightarrow f_{XY} = 3u^2 \times 3v^2 = 9u^2v^2$

$$f_{XY}(u,v) = u+v. \qquad u,v \in [0,1]$$

$$f_X(u) = \int_0^1 (u+v)\,dv = \left[uv + \frac{v^2}{2}\right]\Bigg|_0^1$$

$$= u + \frac{1}{2}$$

$$f_Y(v) = v + \frac{1}{2} \qquad \underset{u+v.}{f_{XY}} \neq f_X f_Y \quad \left(u+\frac{1}{2}\right)\left(v+\frac{1}{2}\right)$$

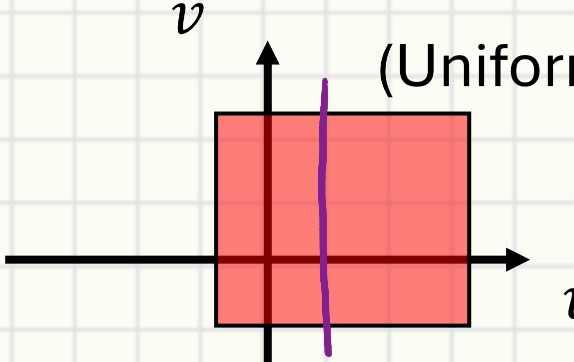
$\Rightarrow$ X & Y are NOT independent.

Slido

X and Y are independent based if $f_{XY}(u, v)$ is...

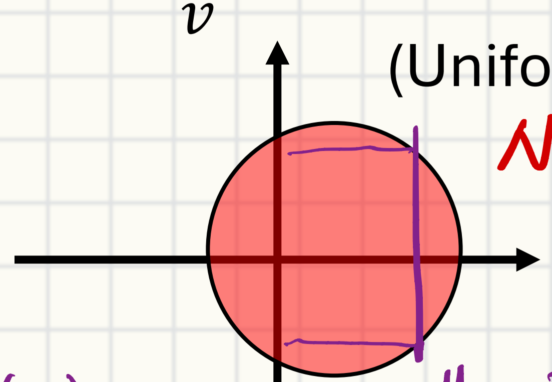
Product set for uniformly dist.

(a)



$\mu_0 \quad f_{Y|X}(v|\mu_0) = f_Y(v)$

(b)



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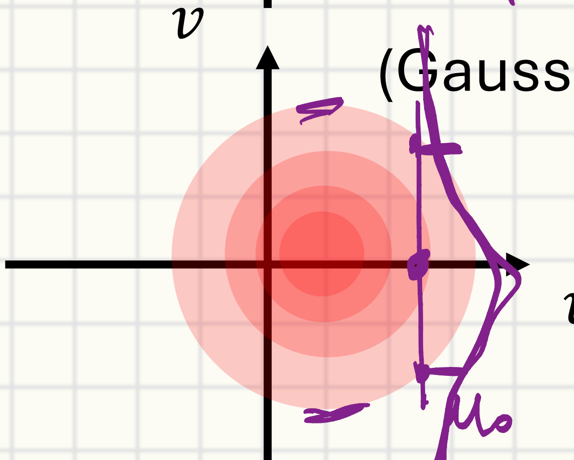
NOT product

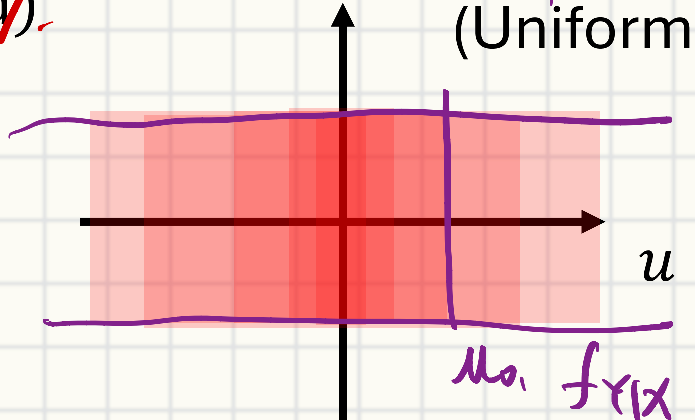
$\mu_0 \quad f_{Y|X} \neq 0$

$[-\sqrt{1-\mu_0^2}, \sqrt{1-\mu_0^2}]$

(c)



(d)



Sums of joint RVs

Motivation

Recall, we learnt if X and Y are independent

- $E[X + Y] = E[X] + E[Y]$
- $\sigma_{X+Y}^2 = \sigma_X^2 + \sigma_Y^2$

What if we know p_{XY} or f_{XY}?

- $E[X + Y] = E[X] + E[Y]$ still holds
- What's the sum of your midterm #1 and midterm #2

Sums of Discrete RVs

Let $S = X + Y$

- $p_S(k) = \sum_{j} P_{XY}(j, k-j)$

  $\hookrightarrow$ X outcome

If X and Y are independent, $p_{XY}(j, k-j) = p_X(j)p_Y(k-j)$

- $p_S(k) = \sum_{j} P_X(j)\, P_Y(k-j) = P_X * P_Y$
- Denoted as convolution

Example

$p_S(k) = \sum_j p_X(j)\, p_Y(k-2j)$

$S = 2X - Y$ $Y = S - 2X$.

Let $X = Bi(n, p)$ and $Y = Bi(m, p)$. $S = X + Y$. Find $p_S(k)$ if X and Y are independent

- Intuitively, $X + Y$ equals "toss a p Head coin $m+n$ times"

  $S \sim Bi(m+n, p)$

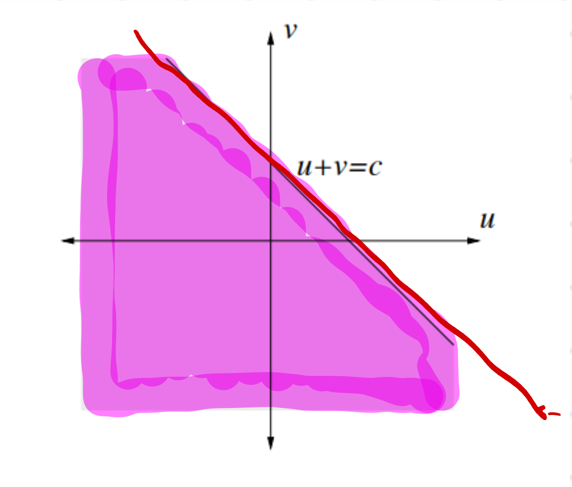
- Verify with formula
- $p_S(k) = \sum_{j=0}^{k} p_X(j) p_Y(k-j)$

$= \sum_{j=0}^{k} \binom{n}{j} p^{j} (1-p)^{n-j} \binom{m}{k-j} p^{k-j} (1-p)^{m-k+j}$

$= \left[\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}\right] p^{k} (1-p)^{m+n-k}$.

$\hookrightarrow \binom{m+n}{k}$

Sums of Continuous RVs



Let $S = X + Y$

- $F_S(c) = P\{S \leq c\} = \int_{-\infty}^{\infty} \int_{-\infty}^{c-u} f_{XY} \, dv \, du$

- $f_S(c) = \boxed{\frac{dF_S(c)}{dc}} = \int_{-\infty}^{\infty} f_{XY}(u, c-u) \, du$

If X and Y are independent

- $f_S(c) =$