Last lecture

Joint PMF (Ch 4.2)

- Definition
- Example

Joint PDF (Ch 4.3)

- Definition
- Example

Agenda

Joint PDF (Ch 4.3)

- Example
  - Uniform distribution
  - Conditional distribution

Independent RV (Ch 4.4)

- From event to RV - CDF
- Check using PDF

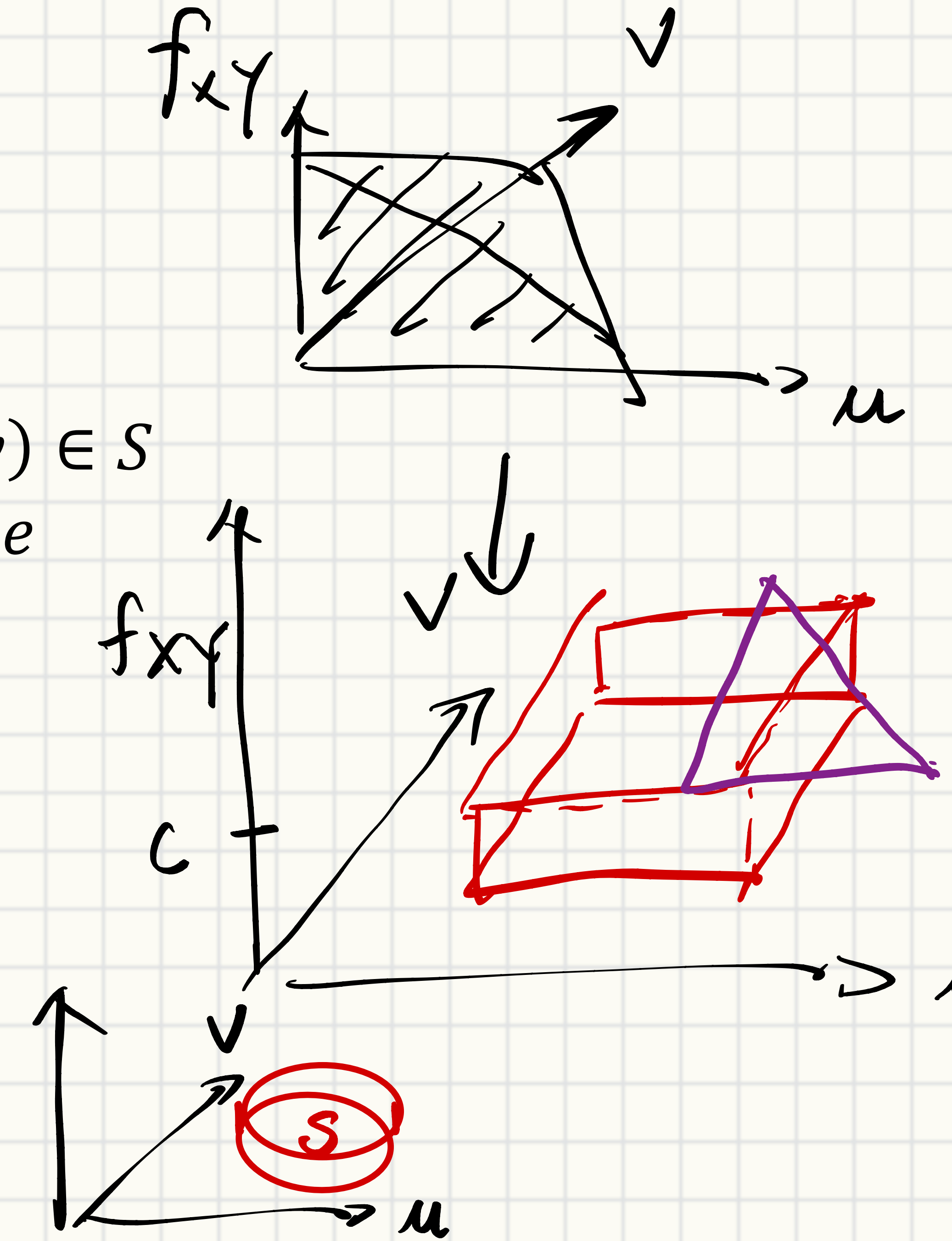
Uniform joint PDF

If pdf are constant over support S

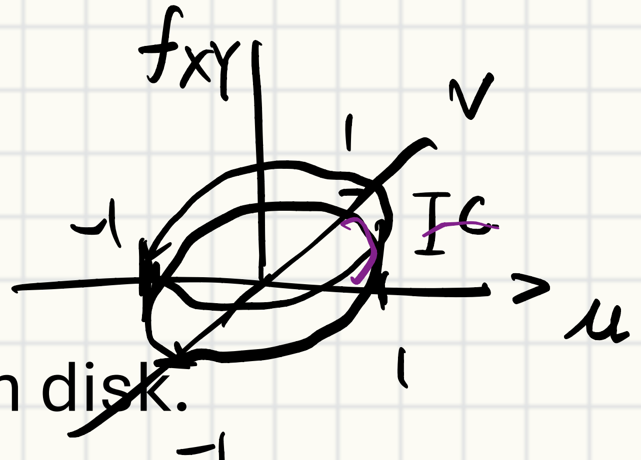
AUC = Area(S) × H = 1

- $f_{X,Y}(u,v) = \begin{cases} \dfrac{1}{area(S)} & if\ (u,v) \in S \\ 0 & else \end{cases}$
- S can be none-rectangular
- $P\{(X,Y) \in A\} = \dfrac{area(A \cap S)}{area(S)}$

→ target area



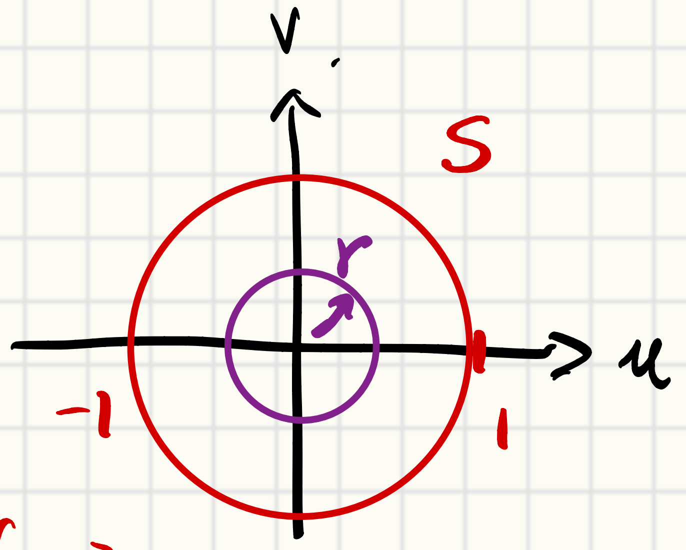
Example



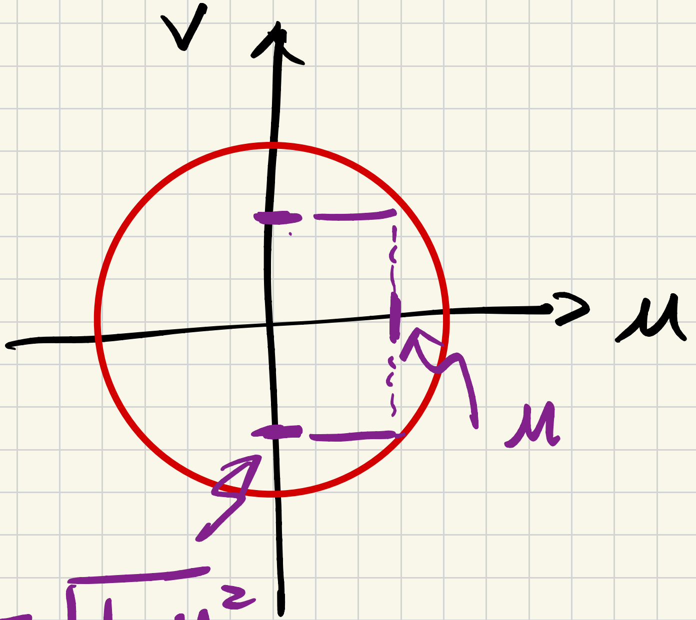
Let (X, Y) uniformly distributed over the uniform disk.

$\text{Area}(S) = \pi$

- $f_{X,Y}(u,v) = \begin{cases} c & if\ u^2 + v^2 \leq 1 \\ 0 & else \end{cases}$, solve
- $c = \dfrac{1}{\text{Area}(S)} = \dfrac{1}{\pi}$
- $P\{X \geq 0 \cap Y \geq 0\} = \dfrac{\frac{1}{4}\pi}{\pi} = \dfrac{1}{4}$
- $P\{X^2 + Y^2 \leq r^2\}$ $\begin{cases} 1 & \text{if } r^2 \geq 1 \\ \dfrac{\text{Area}(A)}{\text{Area}(S)} = \dfrac{r^2}{1} = r^2 & \text{if } r^2 < 1 \end{cases}$
- $f_X(u)$
- $f_{Y|X}(v|u_0)$



$$f_X(u) = \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f_{XY}(u,v)\,dv$$

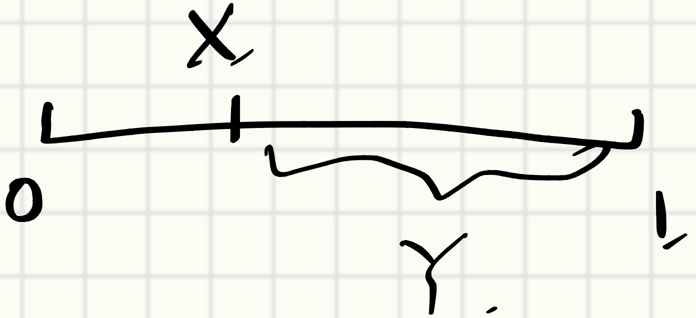


$$= \frac{2\sqrt{1-u^2}}{\pi} \qquad \underline{-1 \le u \le 1}$$

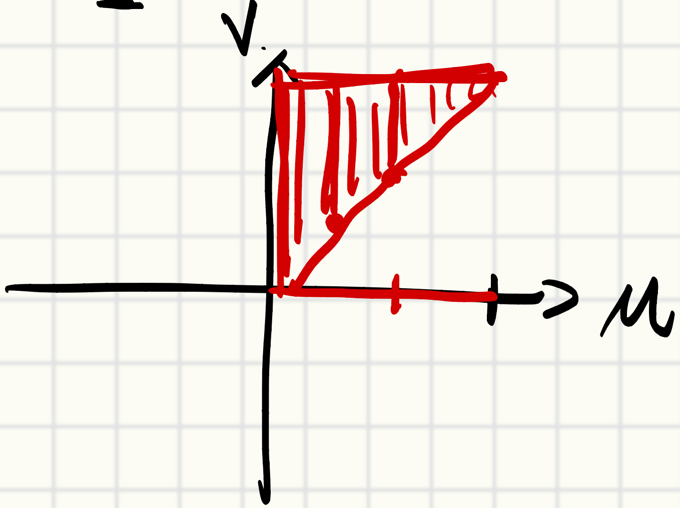
$$f_{Y|X}(v|u_0) = \frac{f_{XY}(u_0,v)}{f_X(u_0)} = \frac{\frac{1}{\pi}}{\frac{2\sqrt{1-u_0^2}}{\pi}} = \frac{1}{2\sqrt{1-u_0^2}}$$

if $-\sqrt{1-u_0^2} \le v \le \sqrt{1-u_0^2}$

Example



Let X uniformly distributed over $[0,1]$. Given $X = u$, Y is uniformly distributed over $[u, 1]$, find



- $f_{Y|X}(v|u_0)$ $= \frac{1}{1-u_0}$ if $u_0 \le v \le 1$
- $f_{XY}(u,v)$

  $= f_{Y|X}(v|u) \cdot \cancelto{1}{f_X(u)}$

  $= f_{Y|X}(v|u) = \frac{1}{1-u}$ if $0 \le u \le v \le 1$
- $f_Y(v)$

$f_Y(v) = \int_0^v \frac{1}{1-u} du = -\ln(1-v)$ 0   else

if $0 < v \le 1$ 0   else

Independent RV

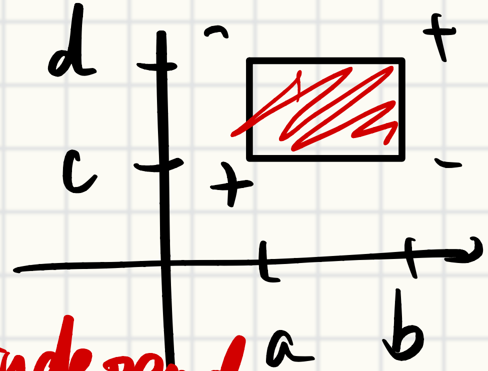
Definition

RV X and Y are independent if for any event pairs $\{X \in A\}$ and $\{Y \in B\}$

- $P\{X \in A, Y \in B\} = P\{X \in A\} \times P\{Y \in B\}$

- Let $A = \{u: u \leq u_0\}$ and $B = \{v: v \leq v_0\}$

$F_{XY}(u_0, v_0) = F_X(u_0) \times F_Y(v_0)$



Even more powerful - for region $R = (a, b] \times (c, d]$ → independent.

- $P\{a < X \leq b, c < Y \leq d\} = F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) + F_X(a)F_Y(c) = (F_X(b) - F_X(a))(F_Y(d) - F_Y(c))$

For PDF and CDF, independent iif

$P_{XY}(u,v) = P_X(u)P_Y(v)$

$f_{XY}(u,v) = f_X(u)f_Y(v)$

Determining independence from PDF

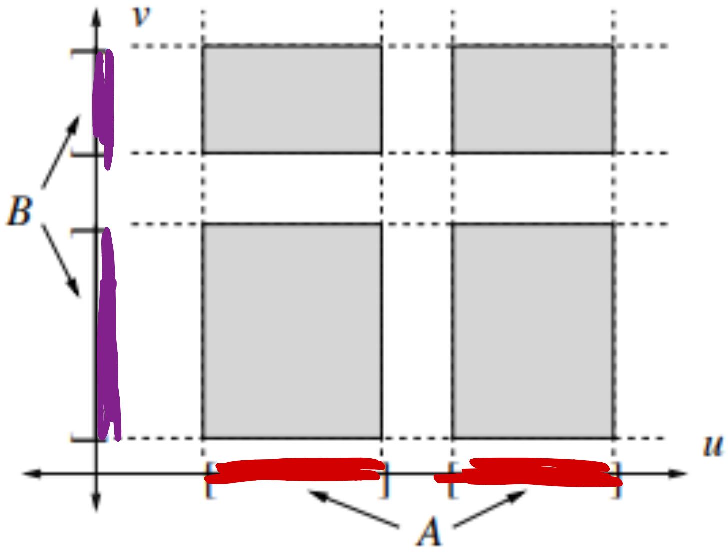
RV X and Y are independent if and only if

- $f_{XY}(u,v) = f_X(u)f_Y(v)$... but others?

Proposition - X and Y are independent if and only if the following condition holds: For all $u \in \mathbb{R}$, either $f_X(u) = 0$ or $f_{Y|X}(v|u) = f_Y(v)$ for all v ∈ R.

$$f_{Y|X} = \frac{f_{XY}}{f_X} = \frac{\cancel{f_X} f_Y}{\cancel{f_X}}$$

Product Set



Let A, B denote a finite union of intervals

- $|A|$ denotes the total length of $|A|$

The product set $A \times B = \{(u, v) : u \in A, v \in B\}$

- The total area $|A \times B| = |A| \times |B|$

Swap property: $S \in \mathbb{R}^2$ has the swap property if

- For any pair of points $(a, b), (c, d) \in S$, (a, d) and (c, b) also in S

Proposition - $S \in \mathbb{R}^2$, S is a product set if and only if it has the swap property

Properties of independent

- If X, Y are independent and jointly continuous type RVs, then support of $f_{X,Y}$ is a product set

$f_{XY}(u, v) > 0$

$f_X(u) > 0$ &

$\to |A|$

$f_Y(v) > 0$

$\to |B|$

- Support X, Y are uniformly distributed over set $S \in \mathbb{R}^2$, then X and Y are independent iif S is a product set