Last lecture

Binary Hypothesis Testing with continuous distribution (Ch 3.10)

- Example

Jointly Distributed RV/ Joint CDF (Ch 4.1)

- Motivation
- Definition
- Properties

Agenda

Joint PMF (Ch 4.2)

- Definition
- Example

Joint PDF (Ch 4.2)

- Definition
- Example

Joint PMF

Roll 2 D6 $\left\langle \begin{matrix} X \\ Y \end{matrix} \right.$ $P_{X,Y}(3,4) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

(and)

If X and Y are discrete, joint PMF $p_{X,Y}(u, v) = P\{X = u, Y = v\}$

$\boxed{\text{Marginalization}}$

- Getting single RV PMF/ PDF from joint PMF/ PDF
- $p_X(u) = \sum_{v_j} p_{X,Y}(u, v_j)$ called "marginal PMF"

↳ PMF of X. → $Y = v_j$ are partitions.

Conditional PMF

- $p_{Y|X}(v|u_0) = \frac{p_{X,Y}(u_0,v)}{p_X(u_0)}$ → $\frac{P\{X = u_0, Y = v\}}{P\{X = u_0\}}$

Example

$Y=3$	0.1	0.1	0
$Y=2$	0	0.2	0.2
$Y=1$	0	0.3	0.1
	$X=1$	$X=2$	$X=3$

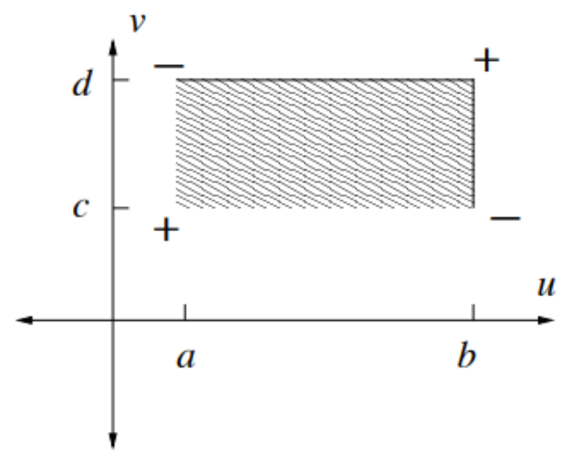
Given joint PMF $p_{X,Y}$ as the table, find

- p_X $\quad P_X(1)=0.1 \quad P_X(2)=0.6 \quad P_X(3)=0.3$
- $p_Y(1,2,3) = [0.4,\ 0.4,\ 0.2]$.
- $P\{X=Y\} = 0+0.2+0 = 0.2$.
- $P\{X>Y\} = 0.3+0.1+0.2 = 0.6$.
- $p_{Y|X}(v|2)$ = "Find PMF of Y given $X=2$"

$$= \frac{P_{XY}(v,2)}{P_X(2)} = \frac{[0.3,\ 0.2,\ 0.1]}{0.6} = [0.5,\ 0.33,\ 0.17]$$

Joint PDF

Joint PDF



If X and Y are continuous, joint PDF $f_{X,Y}(u,v)$ s.t.

- $F_{X,Y}(u,v) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{X,Y}(u,v)dvdu$
- $P\{(X,Y) \in R\} = \int \int_R f_{X,Y}(u,v)dudv$ for piecewise diff. R
- $f_{X,Y}(u,v) \geq 0$ for any $(u,v) \in \mathbb{R}^2$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u,v)dudv =$ 1,

$g(X,Y) = X,$

LOTUS: $E[\underline{g(X,Y)}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \underline{g(u,v)}\,\underline{f_{X,Y}(u,v)}dudv$

- $E[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u \cdot f_{XY}(u,v) \, du \, dv,$

Joint PDF

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\,du\,dv$$

- $E[aX + bY + c] =$ $a\,E[X] + b\,E[Y] + c$

Marginalization - Projection

- $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$
- $E[X] = \int \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv}_{f_X(u)}\, u\,du = \int_{-\infty}^{\infty} u \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\,du$

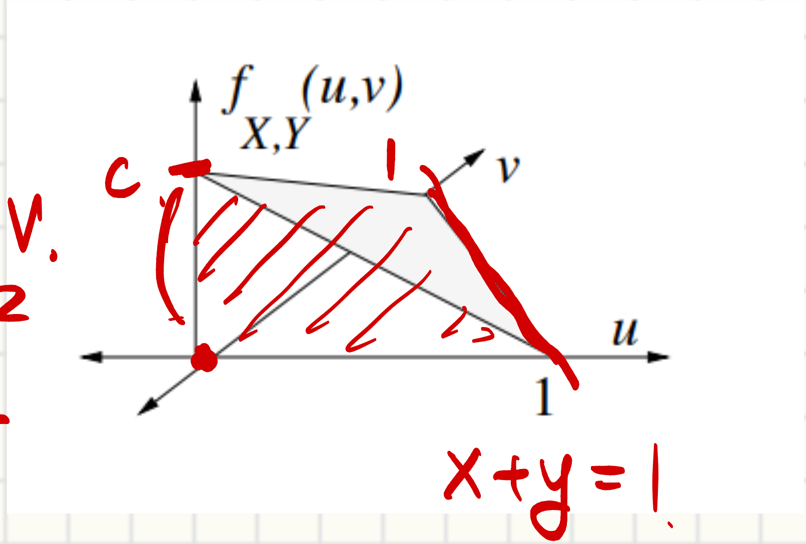
Conditional PDF - Slice

- $f_{Y|X}(v|u_0) = \dfrac{f_{X,Y}(u_0,v)}{f_X(u_0)}$
- Conditional expectation $E[Y|X = u_0] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u_0)\,dv$

Example

Given $f_{X,Y}(u,v) = \begin{cases} c(1-u-v) & if\ u,v \geq 0, u+v \leq 1 \\ 0 & else \end{cases}$, solve

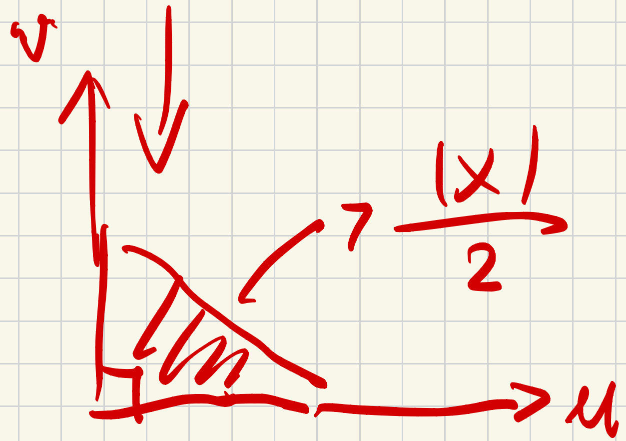
- $c = 6$
- Marginal PDF $f_X(u) = \int_0^{1-u} c(1-u-v)\,dv = \frac{c(1-u)^2}{2}$
- Conditional PDF $f_{Y|X}(v|u_0)$



$f_{Y|X}(v|u_0) = \frac{f_{XY}(u_0, v)}{f_X(u)} = \frac{2(1-u_0-v)}{(1-u_0)^2} \quad 0 \leq v \leq 1-u_0$

$f_{XY} = c(1-u-v)$

Obs. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(u,v)\,du\,dv = \underbrace{1}_{AUC}$



① Geometry

Base $= \frac{1}{2}$. Height $= c$ $\quad V = \frac{\text{Base} \times H}{3} = \frac{c}{6} = 1$

$c = 6$ □

② Integration

$$\int_0^1 \int_0^{1-v} c(1-u-v)\,du\,dv.$$

$$\int_0^1 c\left(u - \frac{u^2}{2} - uv\right)\Bigg|_0^{u=1-v} dv$$

$$= \int_0^1 c\left[(1-v) - \frac{(1-v)^2}{2} - (1-v)v\right] dv$$

$$(1-v)(1-v) = (1-v)^2$$

$$= c\int_0^1 \frac{(1-v)^2}{2}\, dv = c - \frac{(1-v)^3}{6}\Bigg|_0^1 = \frac{c}{6} = 1$$

$$C = 6$$