Last lecture

Generating a customized RV (Ch 3.8.2)

- Intuition $g = F_X^{-1}$
- Examples
  - Uniform to Exponential
  - Uniform to D6 outcome
- Area rule – Compute $E[X]$ using F_X (Ch 3.8.3)

Binary Hypothesis Testing with continuous distribution (Ch 3.10)

- Overview

Agenda

Binary Hypothesis Testing with continuous distribution (Ch 3.10)

- Example

Jointly Distributed RV/ Joint CDF (Ch 4.1)

- Motivation
- Definition
- Properties

Joint PMF (Ch 4.2)

- Definition
- Example

Binary Hypothesis Testing on Continuous Distribution

Overview

Similar to discrete, but with some changes

- $P\{X = u|H_1\} \rightarrow f_1(u)$
- Likelihood Ratio $\Lambda(u) = \frac{f_1(u)}{f_0(u)}$
- LRT rule $\Lambda(X) \begin{cases} > \tau & H_1 \\ < \tau & H_0 \end{cases}$

$\tau_{ML} = 1 \qquad \tau_{MAP} = \frac{\pi_0}{\pi_1}$

$p_{false\ alarm}, p_{miss}, p_e$ remain the same

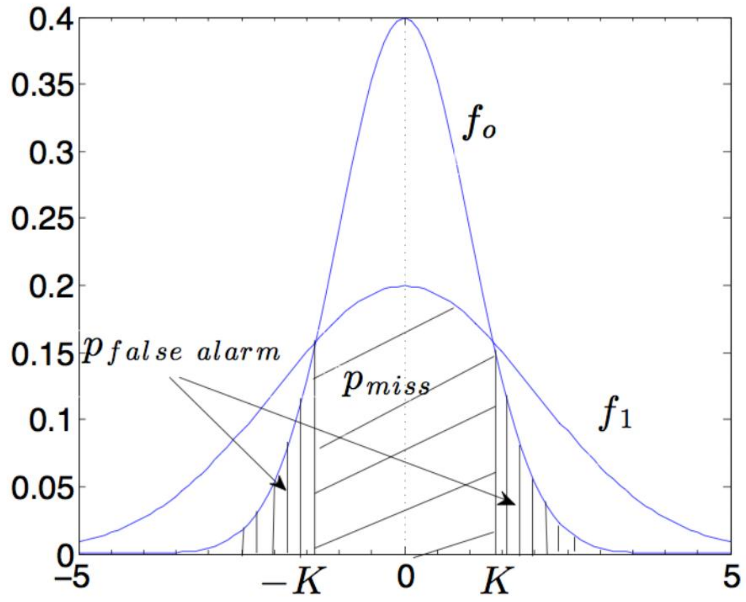
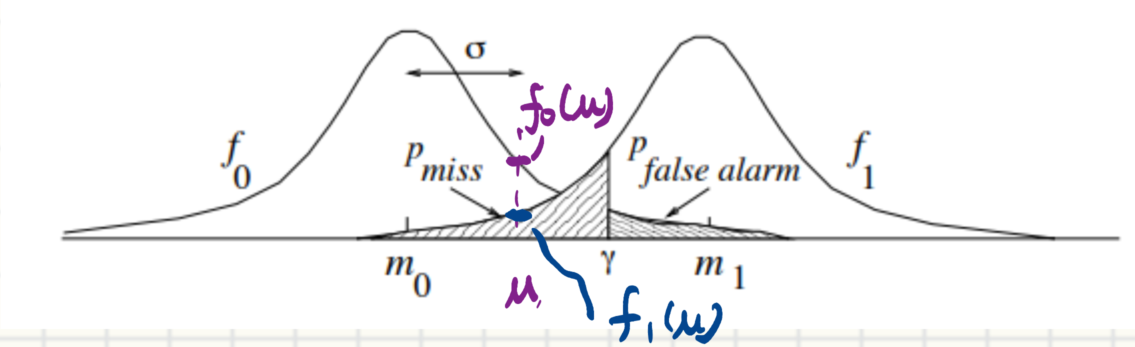


Figure 3.27: $N(0,1)$ and $N(0,4)$ pdfs and ML threshold K.

Example



$(m_1 > m_0)$

X under H_i follows $N(m_i, \sigma^2)$. Given m_i, σ, π_i, Find ML and MAP rule

- $f_i(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(u-m_i)^2}{2\sigma^2}\right\}$
- $\Lambda(u) = \exp\{(u - \frac{m_0+m_1}{2})\boxed{(\frac{m_1-m_0}{\sigma^2})}\}$ $+$
- ML rule –

$\tau \Rightarrow$ Threshold on Λ

$$\frac{f_1}{f_0} = \frac{e^{-\frac{(u-m_1)^2}{2\sigma^2}}}{e^{-\frac{(u-m_0)^2}{2\sigma^2}}}$$

$$= e^{-\left\{\frac{(u-m_1)^2-(u-m_0)^2}{2\sigma^2}\right\}}$$

$\Lambda(u) \geq 1 \quad H_1$

$$u \geq \frac{m_0+m_1}{2}$$

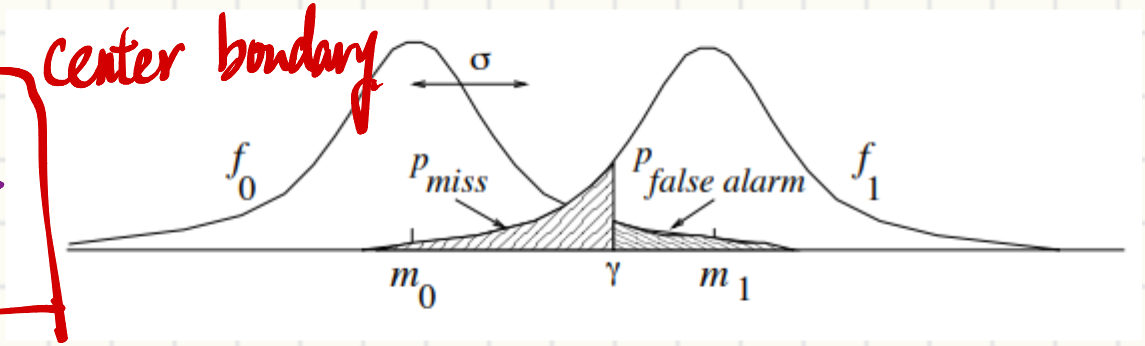
$\Lambda(u)$ is none-decreasing

Threshold on X axis $= \gamma_{ML} = \frac{m_0+m_1}{2}$

Example

$$\left(\frac{\sigma^2}{m_1-m_0}\right)\boxed{\ln\frac{\pi_0}{\pi_1}}+\boxed{\frac{m_0+m_1}{2}}.$$

center boundary



X under H_i follows $N(m_i, \sigma^2)$. Given m_i, σ, π_i, Find ML and MAP rule

- $f_i(u) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(u-m_i)^2}{2\sigma^2}\right\}$
- $\Lambda(u) = \exp\{(u - \frac{m_0+m_1}{2})(\frac{m_1-m_0}{\sigma^2})\}$
- MAP rule – First solve τ.

$$\Lambda(u)\begin{cases} > \tau & H_1 \\ < \tau & H_0 \end{cases}$$

$\downarrow \ln$

$$\ln\Lambda(u)\begin{cases} > \ln\tau & H_1 \\ < \ln\tau & H_0 \end{cases}$$

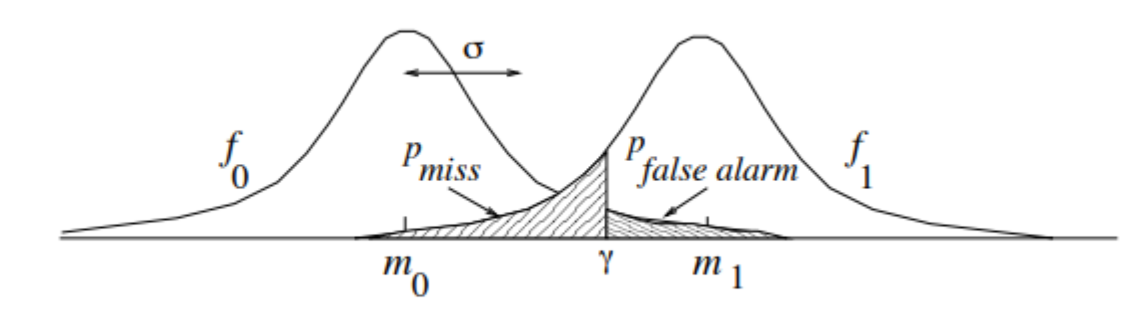
$$\left(x - \frac{m_0+m_1}{2}\right)\left(\frac{m_1-m_0}{\sigma^2}\right) \gtrless \ln \tau. \quad H_1$$

$$\Rightarrow \quad x \gtrless \underbrace{\left(\frac{\sigma^2}{m_1-m_0}\right)\ln\tau + \frac{m_0+m_1}{2}}_{\gamma.}$$

$$\text{MAP} \Rightarrow \quad \tau = \frac{\pi_0}{\pi_1}$$

$$\gamma_{MAP} = \left(\frac{\sigma^2}{m_1-m_0}\right)\ln\frac{\pi_0}{\pi_1} + \frac{m_0+m_1}{2}.$$

Example



X under H_i follows $N(m_i, \sigma^2)$. Given m_i, σ, π_i, Find ML and MAP rule

- $f_i(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(u-m_i)^2}{2\sigma^2}\right\}$
- $\Lambda(u) = \exp\{(u - \frac{m_0+m_1}{2})(\frac{m_1-m_0}{\sigma^2})\}$
- $P_{miss} =$ $P(X < \gamma \mid H_1) = P\left(\frac{X-m_1}{\sigma} < \frac{\gamma - m_1}{\sigma} \middle| H_1\right)$

  $= Q\left(\frac{m_1-\gamma}{\sigma}\right)$
- $P_{false\ alarm} =$

  $Q\left(\frac{\gamma - m_0}{\sigma}\right)$

Slido

$f_1 = Unif([2,6]), f_0 = Unif([0,3])$. Assume $\frac{\pi_1}{\pi_0} = \frac{3}{4}$

- Using MAP rule, the region we claim H_1

$\hat{x}$

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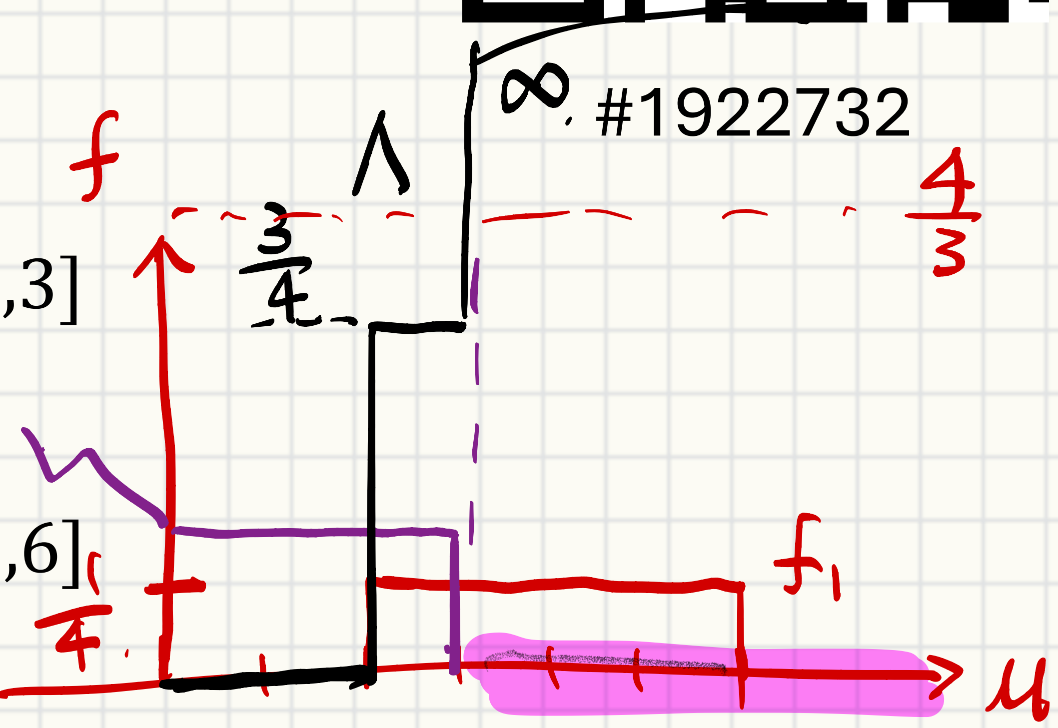
(a) $None$

(b) $[2,3]$

(c) $[2,6]$

(d) $[3,6]$



$\Lambda > \tau_{MAP} = \boxed{\frac{4}{3}}$

Jointly Distributed Random Variables

Motivation

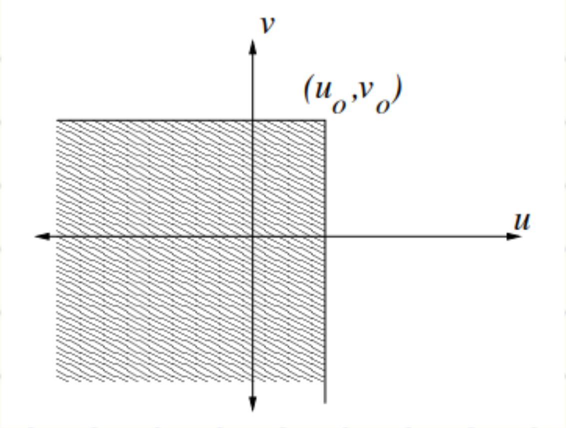
Given X and Y, we have learnt

- Independence
- Function & Scaling (e.g. X=3Y-2)

But real-world cases are more complex

- How to show the " correlation / coexistence " between X and Y
- "Jointly distributed" RVs
- $F_X(u) \quad \rightarrow \quad F_{X,Y}(u,v)$
- $P_X(u) \quad \rightarrow \quad P_{X,Y}(u,v)$

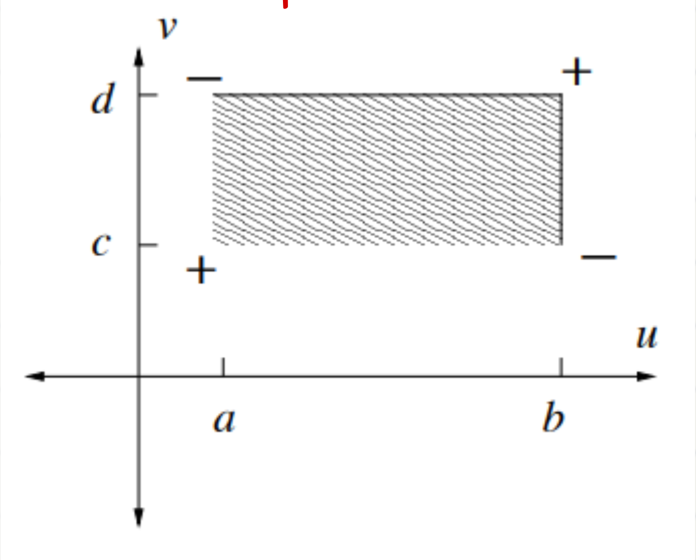
Joint CDF



AND

Joint CDF of X and Y

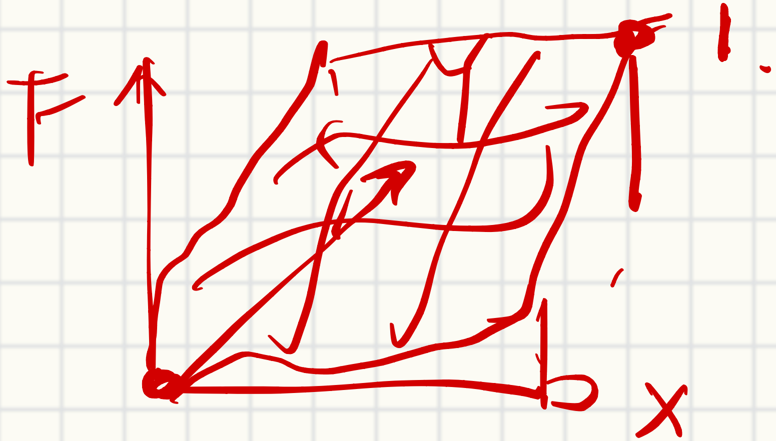
- $F_{X,Y}(u, v) = P\{X < u, Y < v\}$ for any $(u, v) \in \mathbb{R}^2$
- Completely defines all events concerning X and Y
- For a 2D rectangle region $R = (a, b] \times (c, d]$
  - $P\{\underbrace{(X, Y) \in R}_{a \leq X \leq b,\ c \leq Y \leq d}\} =$ $F_{XY}(b,d) - F_{XY}(a,d) - F_{XY}(b,c) + F_{XY}(a,b)$

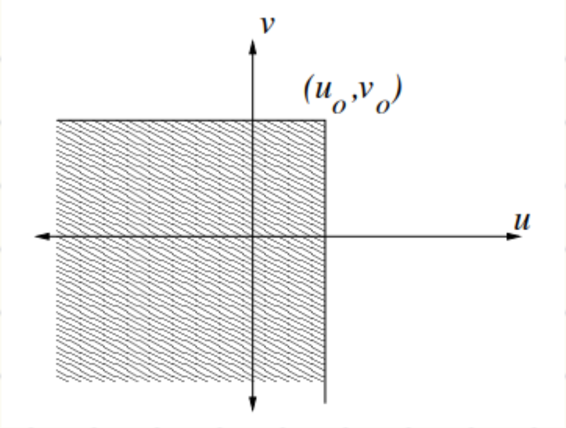


- $F_X(u) = F_{X,Y}(u, \infty)$

$F_Y(v) = F_{XY}(\infty, v)$

Joint CDF Properties





Denote $F_{X,Y}$ as F

- $0 \le F(u, v) \le 1$ for all $(u, v) \in \mathbb{R}^2$
- along u and along v respectively
  - F is none decreasing
  - F is right-continuous
- $\lim_{u \to -\infty} F(u, v) = 0$
- $\lim_{u \to \infty} \lim_{v \to \infty} F(u, v) = 1$