Last lecture

Functions of a random variable (Ch 3.8)

- Examples for
  - Case by case
  - g is cosine/ tangent
- General form of increasing function g
- Create uniform distribution using CDF as a function

$$U = F_X(X)$$

Agenda

Generating a customized RV (Ch 3.8.2)

- Intuition $g = F_X^{-1}$
- Examples
  - Uniform to Exponential
  - Uniform to D6 outcome
- Area rule – Compute $E[X]$ using F_X (Ch 3.8.3)

Binary Hypothesis Testing with continuous distribution (Ch 3.10)

- Overview
- Example

Generating a customized RV

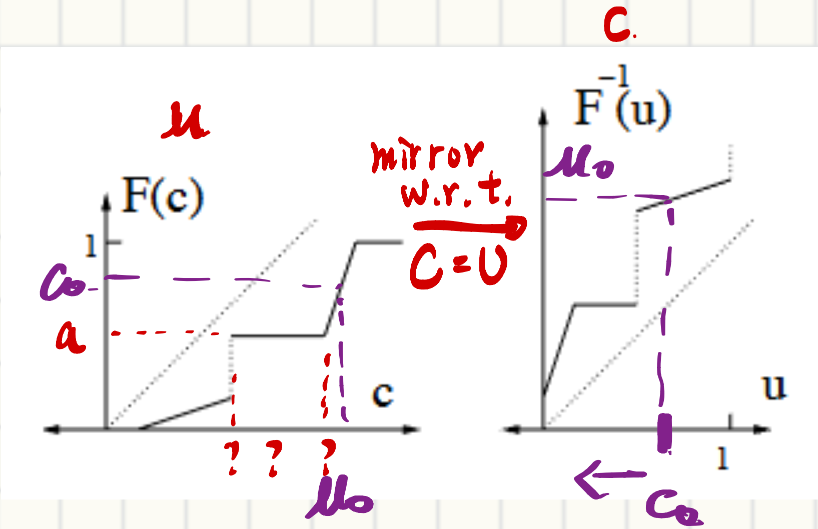
If we want to generate (sample) $X = g(U \sim Uni([0,1]))$ to fit F_X

- $U = F_X(X) \leftarrow$ Apply for all X.
- $F_X^{-1}(U) = X$
- $g = F_X^{-1}$, but F_X^{-1} may not be a function
  - $F_X(c_1) = F_X(c_2)$
- For any F, define

$$F^{-1}(u) = \min\{c: F(c) \geq u\}$$

- $F: c \to u, \qquad F^{-1}: u \to c$

X support.



$F^{-1}(u_0) \leq c_0$ if and only if $u_0 \leq F(c_0)$

- $F_X(c) = P\{F^{-1}(U) \leq c\} = P\{U \leq F(c)\} = F(c)$

Example – Uniform to Exponential

Find g s.t. $g(U) \sim Exp(\lambda = 1)$ when $U \sim Uniform([0,1])$

- $F_T(c) = 1 - e^{-c} = u$ for $c \geq 0$ $\quad F_T(t) = 1 - e^{-\lambda t}$.
- Find $g(u) = F^{-1}(u)$ in terms of u

- $1 - e^{-c} = u$ $\quad \Rightarrow c \to u. \quad F_X$
- $c = -\ln(1-u)$ $\quad \Rightarrow u \to c \quad F_X^{-1}$
- $g(u) = F^{-1}(u) = -\ln(1-u)$ for $0 \leq u \leq 1$
- Check, if $c \geq 0$ $\Rightarrow$ support of Exp.
  - $\underbrace{P\{-\ln(1-U) \leq c\}}_{g(U)}$ $= P\{\ln(1-U) \geq -c\}$
    $= P\{1 - U \geq e^{-c}\}$
    $= P\{U \leq 1 - e^{-c}\} = F(c)$

Example – Uniform to Exponential

Find g s.t. $g(U) \sim Exp(\lambda = 1)$ when $U \sim Uniform([0,1])$

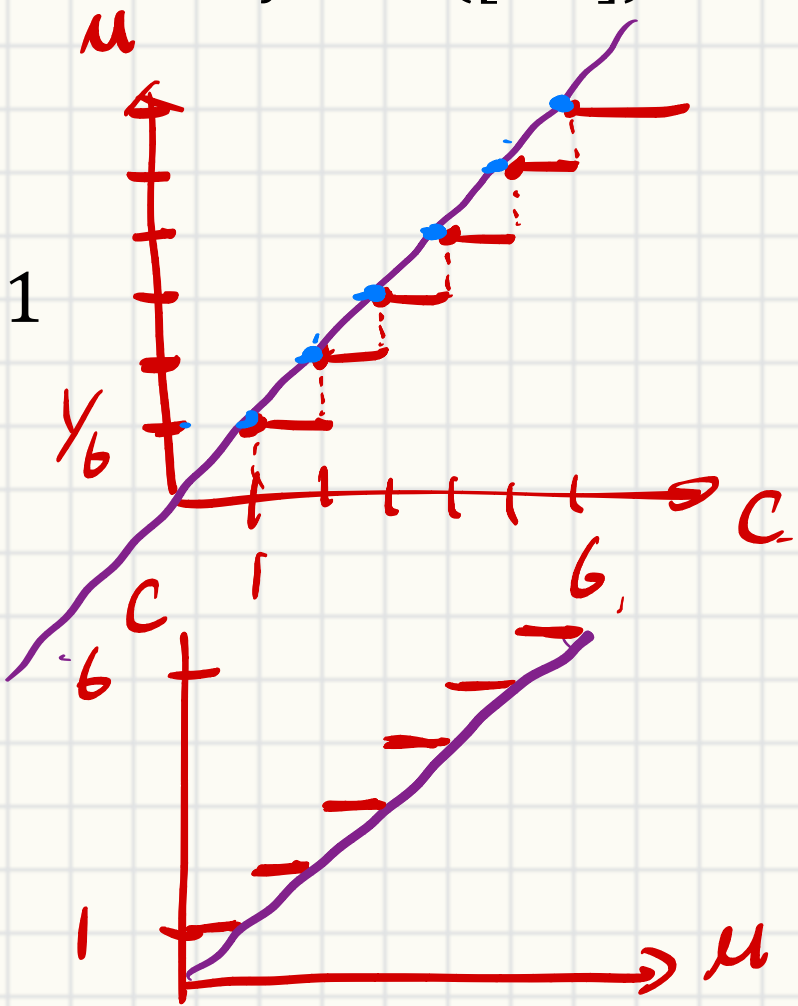
- $F(c) = 1 - e^{-c} = u$ for $c \geq 0$
- Find $g(u) = F^{-1}(u)$ in terms of u

- $1 - e^{-c} = u$
- $c = -\ln(1 - u)$
- $g(u) = F^{-1}(u) = -\ln(1 - u)$ for $0 \leq u \leq 1$
- Note - g is not unique
  - E.g. U and $1 - U$ are all uniform, so $-\ln u$ is also a valid g

Example – Uniform to Dice outcome

Find g s.t. $g(U) \sim \{$“Fair die distribution”$\}$ when $U \sim Uniform([0,1])$

- $F(i) = \frac{\lfloor i \rfloor}{6} = u$ for $1 \leq i \leq 6$
- $g(u) = F^{-1}(u)$
- $g(u) = i$ where $\frac{i-1}{6} \leq u \leq \frac{i}{6}$ for $0 \leq u \leq 1$



The area rule for expectation based on the CDF

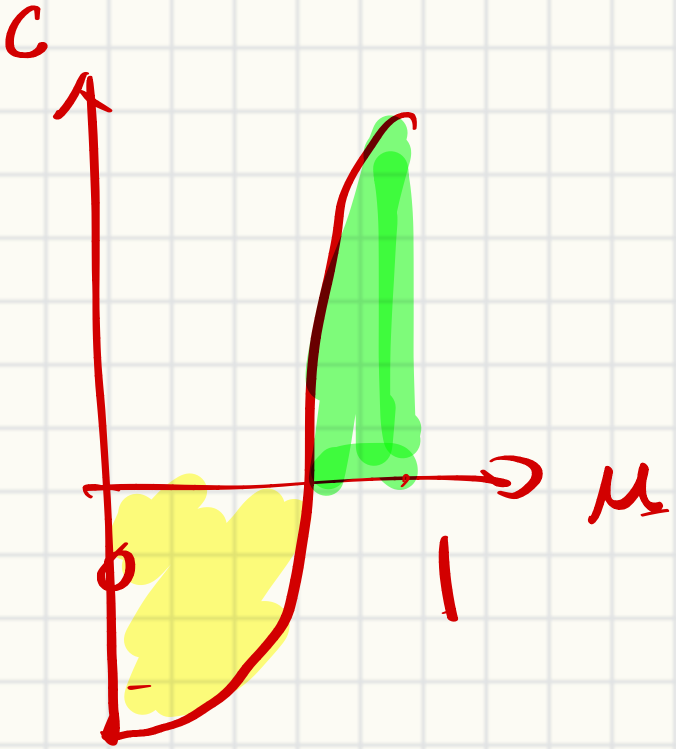
$$E[X] = Area^{+} - Area^{-} = \int_0^{\infty} \big(1 - F_X(c)\big)dc - \int_{-\infty}^{0} F_X(c)dc$$

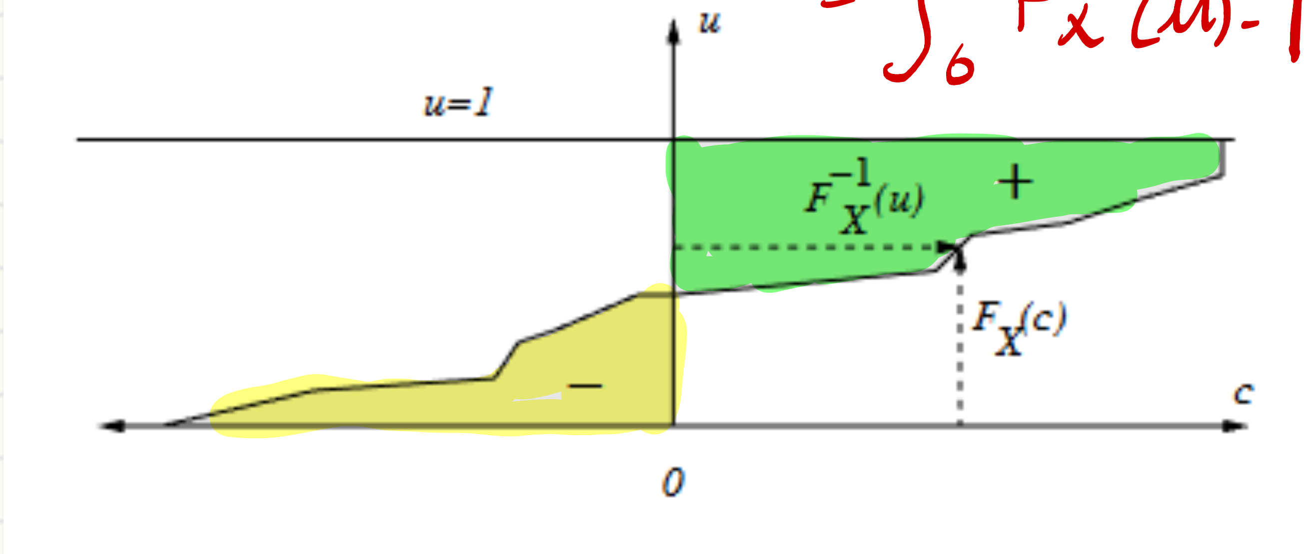
$$E[X] = E[F_X^{-1}(U)] = \int_0^1 F_X^{-1}(u)du$$

$P_U(u)$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= \int_0^1 F_X^{-1}(u) \cdot 1\, du$$





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Order the $E[X]$ from high to low   A C B D

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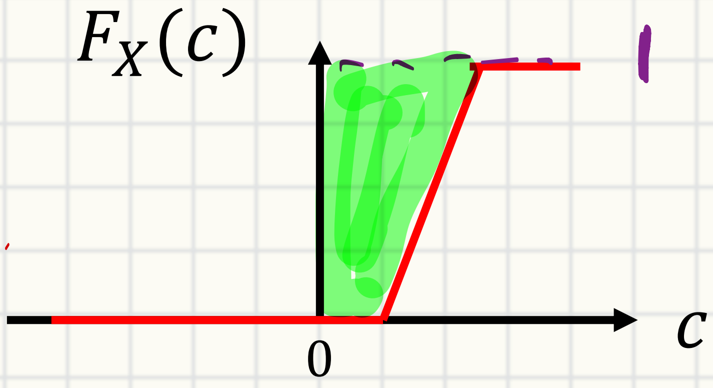
(A)



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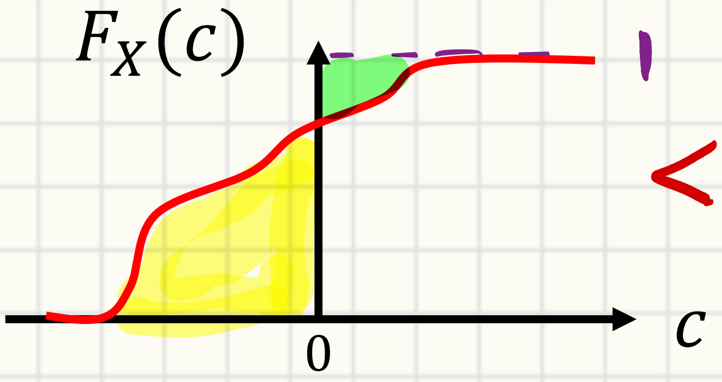

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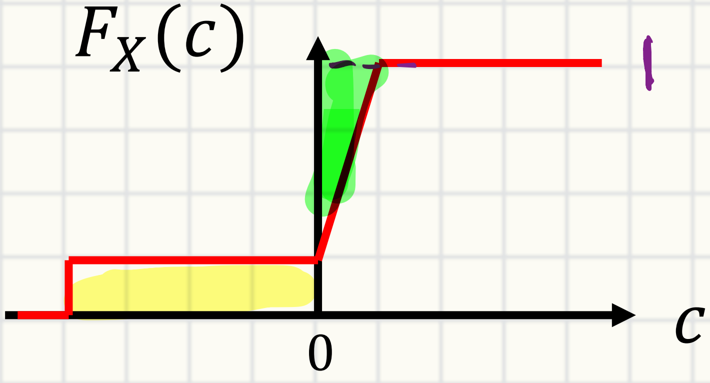

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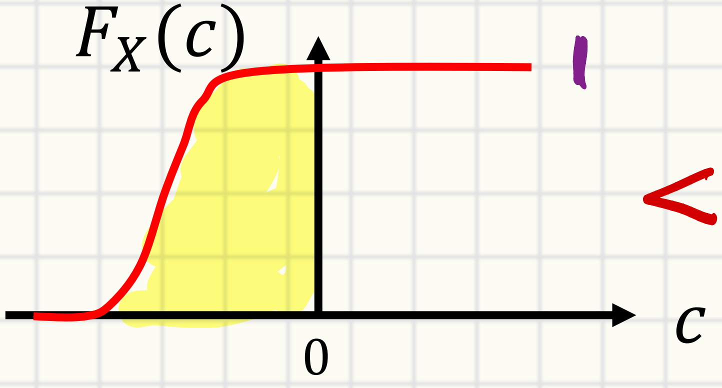

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Binary Hypothesis Testing on Continuous Distribution

Overview

condition on H_1

Similar to discrete, but with some changes

- $P\{X = u | H_1\} \rightarrow f_1(u)$
- Likelihood Ratio $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)}$
- LRT rule $\Lambda(X) = \begin{cases} > \tau & H_1 \\ < \tau & H_0 \end{cases}$

$p_{false\ alarm}, p_{miss}, p_e$ remain the same

likelihood function

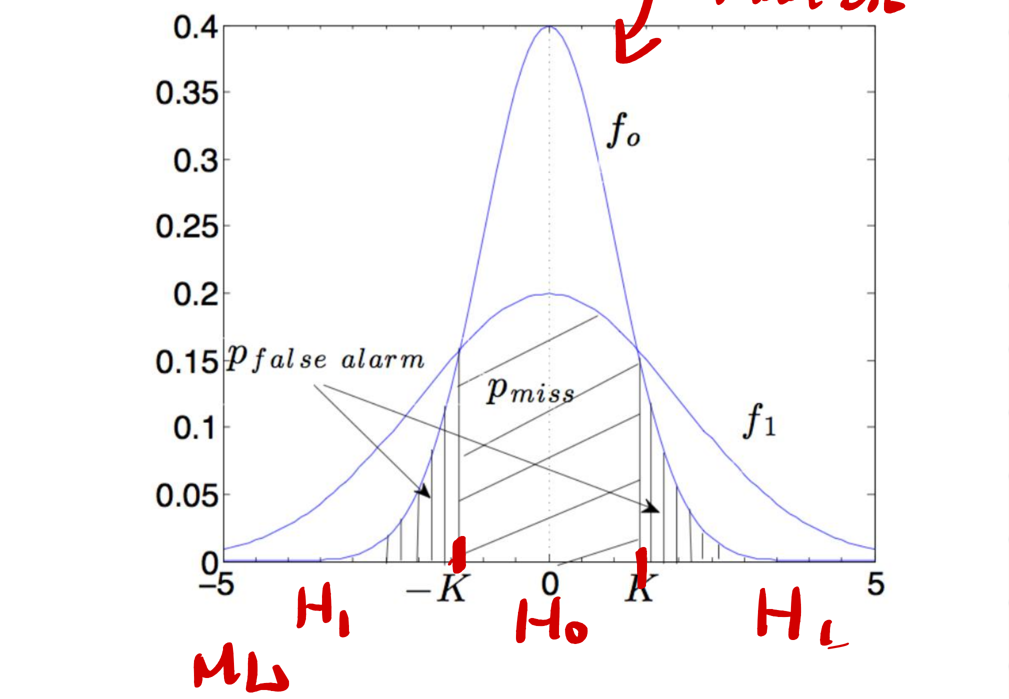

Figure 3.27: $N(0,1)$ and $N(0,4)$ pdfs and ML threshold K.