Agenda

Final Review I (Post-midterm2)

- BHT
- Joint Probability
- Estimators
- Functions of RV

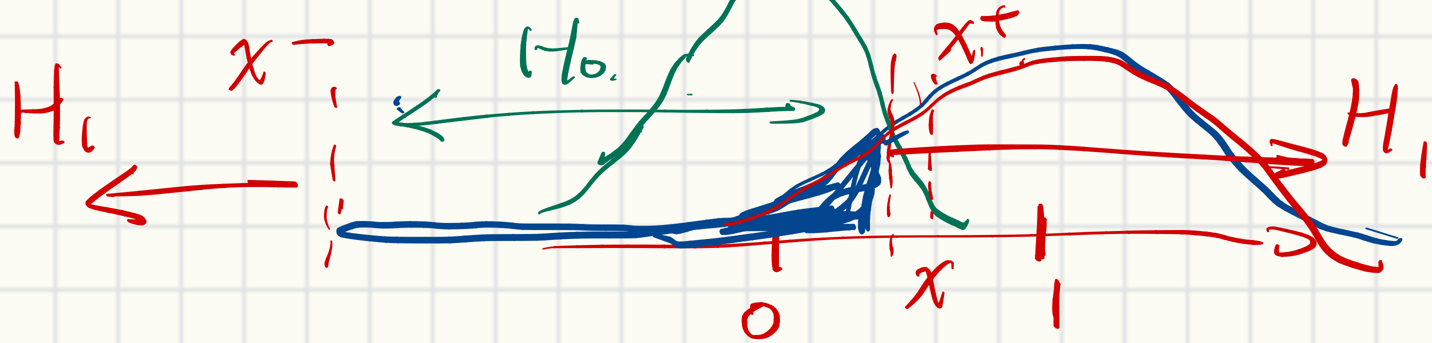
Binary Hypothesis Testing

	$X=1$	$X=2$
H_1	$f_1(1)$	$f_1(2)$
H_0	$f_0(1)$	

$\Lambda(1)$

- Likelihood matrix/ function
  - $f_i(k) = P\{X = k|H_i\}$
  - Likelihood Ratio $\Lambda(k) = \dfrac{f_1(k)}{f_0(k)}$
  - ML method - $f_1(k)$ $vs.$ $f_0(k)$ or $\Lambda(k) > 1$?

- Joint Probability Matrix $P(X_i H_i)$ (function)
  - MAP method - $\Lambda(k) > \tau_{MAP} = \dfrac{\pi_0}{\pi_1} \rightarrow P(H_0)$



Example -

Let $f_1 \sim N(\mu = 1, \sigma^2 = 2)$ and $f_0 \sim N(\mu = 0, \sigma^2 = 1)$

- Find ML rule and MAP rule corresponding $P_{miss}, P_{false\ alarm}$

$$f_1(x) = \frac{1}{\sqrt{2\pi 2}} \exp\left(-\frac{(x-1)^2}{2\times 2}\right)$$

$$f_0(x) = \frac{1}{\sqrt{2\pi\times 1}} \exp\left(-\frac{x^2}{2}\right)$$

$$\Lambda(x) = \frac{f_1(x)}{f_0(x)} = \frac{1}{\sqrt{2}} \exp\left(\boxed{\frac{2x^2 - (x-1)^2}{4}}\right) > 1,$$

$$\exp(\cdot) > \sqrt{2}$$

$$\frac{2x^2 - (x-1)^2}{4} > \frac{\ln 2}{2}$$

$$\Downarrow$$

$$x > x^+$$

$$x < x^-$$

$$P_{miss} = P\{\text{Claim } H_0 \mid H_1\}$$

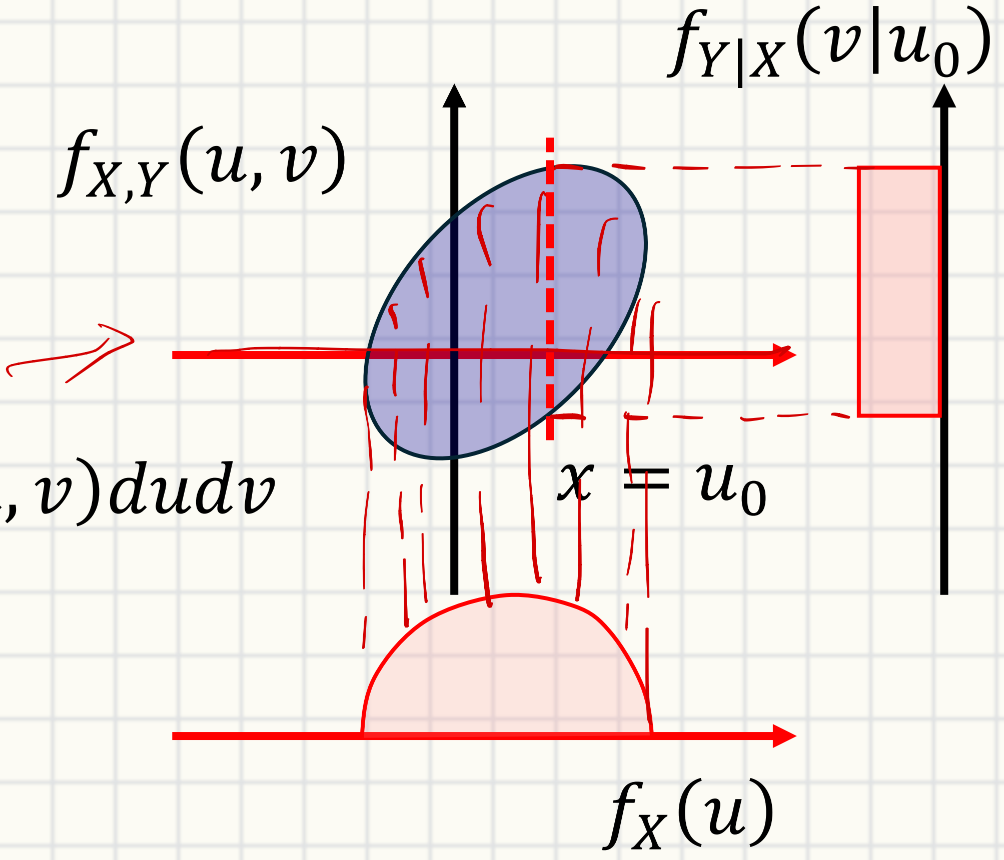
$$Y \sim \underline{N(1, 2)}$$

$$= P\{x^- \le Y \le x^+\}$$

$$= P\left\{\frac{x^- - 1}{\sqrt{2}} \quad \frac{Y-1}{\sqrt{2}} \le \frac{x^+ - 1}{\sqrt{2}}\right\}$$

$$= \Phi\left(\frac{x^+ - 1}{\sqrt{2}}\right) - \Phi\left(\frac{x^- - 1}{\sqrt{2}}\right)$$

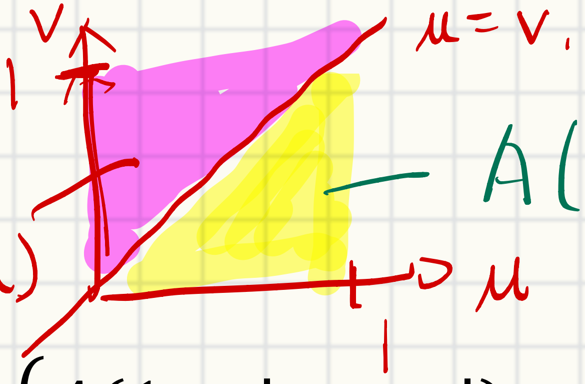
Joint Probability Density



X and Y are described by $f_{X,Y}(u,v)$

- $E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v)\,du\,dv$
  - Double integrate over x and y
- $f_{Y|X}(v|u_0) = \frac{f_{X,Y}(u_0,v)}{f_X(u_0)}$
  - Conditional distribution
  - Useful in **unconstrained estimator** $E[Y|X] = \int y f_{Y|X}\,dy$
- $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv$ and $E[X] = \int_{-\infty}^{\infty} u \int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\,du$
  $= \int_{-\infty}^{\infty} u f_X(u)\,du$
  - Marginalization

Example



$A(1-u+v)$

$A(1-v+u)$

$$f_{X,Y}(u,v) = \begin{cases} A(1-|u-v|) & if \quad 0<u<1, 0<v<1 \\ 0 & else \end{cases}$$

- Solve A

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(u,v) = 1$$

$$\hookrightarrow \int_0^1\int_0^u A(1-u+v)\,dv\,du \; +$$

$$\int_0^1\int_0^v A(1-v+u)\,du\,dv = 1.$$

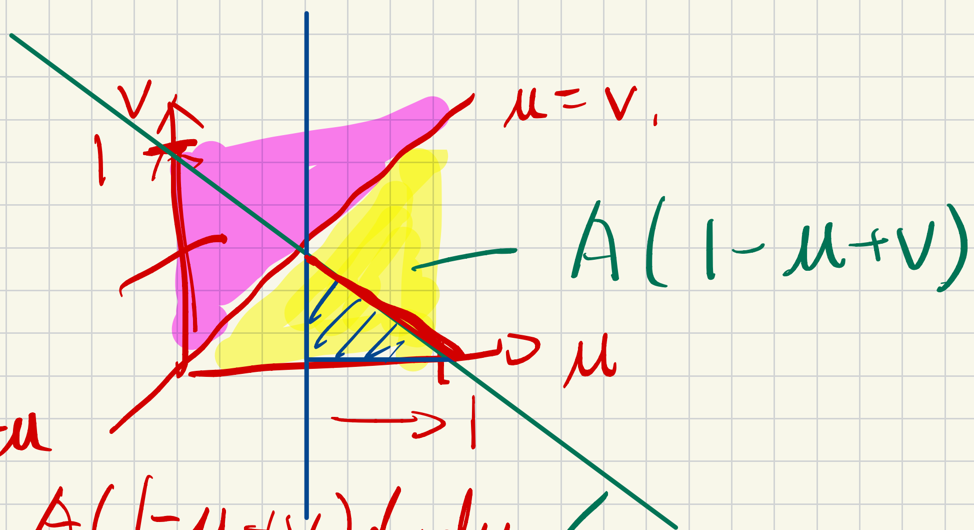
$$\Rightarrow A = \frac{3}{2}.$$

- Find $f_X(u)$ and $f_Y(v)$

- Find $P\{X+Y<1|X>\frac{1}{2}\}$

$$\int_{-\infty}^{\infty} f_{XY}(u,v)\,dv = \int_0^u \frac{3}{2}(1-u+v)\,dv + \int_u^1 \frac{3}{2}(1-v+u)\,dv$$

$$= \frac{3}{2}\left[-u^2+u+\frac{1}{2}\right]$$



$$P\left\{X+Y<1 \;\middle|\; X>\frac{1}{2}\right\}$$

$$= \frac{P\{X+Y<1,\ X>\frac{1}{2}\}}{P\{X>\frac{1}{2}\}} = \frac{\int_{\frac{1}{2}}^1 \int_0^{1-u} A(1-u+v)\,dv\,du}{\int_{\frac{1}{2}}^1 u\, f_X(u)\,du} = \frac{3}{16}$$

Estimators

- Constant estimator
  - $c^* = E[Y]$, MSE = $Var(Y)$

- Unconstraint estimator
  - $g^*(X) = E[Y|X]$ MSE=$E[Y^2] - E[(E[Y|X])^2] = E[Y^2] - E[\left(g^*(X)\right)^2]$
  - Best estimator, but requires $f_{Y|X}$

- Linear estimator
  - $\hat{E}[Y|X] = \mu_Y + \frac{Cov(X,Y)}{Var(X)}(X - \mu_X) = \mu_Y + \rho_{X,Y}\sigma_Y(\frac{X-\mu_X}{\sigma_X})$
  - MSE= $\sigma_Y^2 - \frac{\left(Cov(X,Y)\right)^2}{Var(X)} = \sigma_Y^2(1 - \rho_{X,Y}^2)$

Example

$$\iint uv\,(u+v)\,du\,dv \qquad E[XY] - \mu_X \mu_Y$$

$$f_{X,Y}(u,v) = \begin{cases} c(u+v) & if \quad 0 < u < 1, 0 < v < 1 \\ 0 & else \end{cases}$$

$$f_X(u) = \int_0^1 (u+v)\,dv = u + \frac{1}{2}$$

- Find c^* for Y $\qquad c = 1$
- Find $E[Y|X]$

$$f_{Y|X} = \frac{f_{XY}(u,v)}{f_X(u)} = \frac{u+v}{u+\frac{1}{2}} = \frac{2u+2v}{2u+1}$$

- Find $\hat{E}[Y|X]$

$$E[Y|X] = \int v\,\frac{(2u+2v)}{2u+1}\,dv$$

μ_X, μ_Y, $Cov(X,Y)$, $Var(X)$

Functions of RVs

Given $f_X(u) \longrightarrow f_Y(v)$

- Single RV $Y = g(X)$
  1. Find support of X and Y
  2. Find $F_Y(c)$ from integrating $f_X(x)$ over $\{x: g(x) \leq c\}$
  3. Get $f_Y = F_Y{}'$

$\Rightarrow f_Y = \frac{d}{dY} \int_0^{h(x)} f_X(u)\, du \rightarrow F_Y(c)$

- Multiple RV $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- $f_{W,Z}(\alpha, \beta) = \frac{1}{|\det(A)|} f_{X,Y}(u, v) = \frac{1}{|\det(A)|} f_{X,Y}(A^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix})$

Example

$W = 2X - Y, Z = X + 2Y$. Express $f_{W,Z}(\alpha, \beta)$ in terms of $f_{X,Y}$

- $\begin{pmatrix} W \\ Z \end{pmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$

- $Det(A) =$

- For $(W, Z) = (\alpha, \beta)$, $(X, Y) =$

- $f_{W,Z}(\alpha, \beta) =$