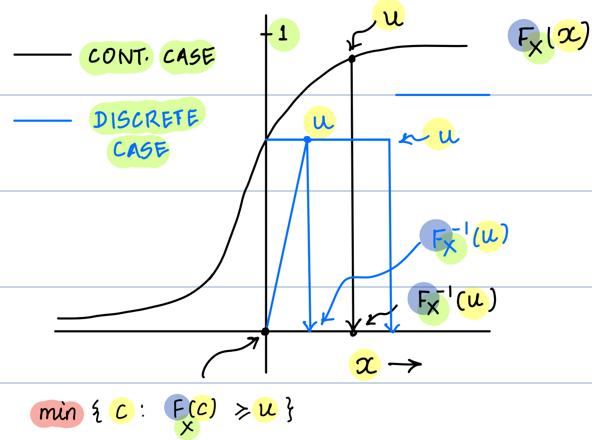
LECTURE 27 : PROBABILITY INTEGRAL TRANSFORM AND GENERATING A RV WITH A SPECIFIED DISTRIBUTION

- TOPICS TO COVER (BASED ON CH 3.7–3.8)
  - → PROBABILITY INTEGRAL TRANSFORM
  - → GENERATING A RV WITH A SPECIFIED DISTRIBUTION



→ PROBABILITY INTEGRAL TRANSFORM

X IS A CONT-TYPE RV WITH CDF F_X E.G. $X \sim \text{EXP}(1)$ $F_X(x) = 1 - e^{-x}$ $Y := F_X(X) = 1 - e^{-X}$

DEFINE $Y := \overset{g(\cdot)}{F_X}(X)$ THEN $Y \sim \text{UNIFORM}[0,1]$

E.G. $X \sim U[0,1]$ $F_X(x) = x$

$Y := F_X(X) = X \sim U[0,1]$

PROOF: $F_Y(c) := P\{Y \leq c\}$

$= P\{F_X(X) \leq c\}$

HOW?

$= P\{X \leq F_X^{-1}(c)\}$ EXISTS AS F_X INCREASES CONTINUOUSLY FROM 0 TO 1

$= F_X(F_X^{-1}(c))$

$= c$: DISTRIBUTION FUNCTION OF A UNIFORM$[0,1]$ RV

$\Rightarrow Y \sim U[0,1]$

→ GENERATING A RV WITH A SPECIFIED DISTRIBUTION

QUESTION: GIVEN A UNIFORM $[0,1]$ RV, HOW TO GENERATE/SIMULATE A RV $X \sim F_X$?

SOLUTION: DEFINE $F_X^{-1}(u) := \min\{c : F_X(c) \geq u\}$ $u \in [0,1]$.

THEN $Y := F_X^{-1}(U) \sim F_X$. E.G. $X \sim \text{EXP}(1)$ $F_X(x) = 1 - e^{-x}$ $Y := F_X^{-1}(U)$

PROOF: $F_Y(c) = P\{Y \leq c\}$

$= P\{F_X^{-1}(U) \leq c\}$

$$= P\{ U \le F_X(c) \} = F_X(c)$$

$\Rightarrow$ $Y := F_X^{-1}(U)$ HAS THE SAME CDF AS X.

HENCE, GIVEN $U \sim$ UNIFORM $[0,1]$ AND A DF F_X, GENERATE A RV FROM F_X AS

$$X := F_X^{-1}(U)$$

EXAMPLE: $U \sim$ UNIFORM$[0,1]$, FIND A $g(\cdot)$ SUCH THAT $g(U) \sim$ EXPONENTIAL(1).

SOLUTION: $X \sim$ EXPONENTIAL(1) : $F_X(c) = \begin{cases} 1 - e^{-c}, & c \geq 0, \\ 0, & \text{OTHERWISE}. \end{cases}$

DEFINE $Y := F_X^{-1}(U) \sim$ EXPONENTIAL(1)

$\therefore$ $g(U) := F_X^{-1}(U)$

TO OBTAIN THE INVERSE FUNCTION:

SET $u = F_X(c)$ AND SOLVE FOR c!

$$\Rightarrow \quad u = 1 - e^{-c}$$

$$\Rightarrow \quad 1 - u = e^{-c}$$

$$\Rightarrow \quad \ln(1-u) = -c$$

$$\Rightarrow \quad c = -\ln(1-u)$$

$$\Rightarrow \quad g(u) = -\ln(1-u)$$

$\therefore$ GIVEN $U \sim$ UNIFORM $[0,1]$, $-\ln(1-U) \sim$ EXPONENTIAL(1)

EXAMPLE : FIND A g SUCH THAT, IF $U \sim \text{UNIFORM}[0,1]$,

$g(U) \sim$ DISTRIBUTION OF THE NUMBER SHOWING FOR THE EXPERIMENT OF ROLLING A FAIR DIE

SOLUTION : WE FIRST UNDERSTAND THE DISTRIBUTION OF THE NUMBER SHOWING FOR THE EXPERIMENT OF ROLLING A FAIR DIE :

X : THE NUMBER SHOWING FOR THE EXPERIMENT OF ROLLING A FAIR DIE

$$P\{X = i\} = \begin{cases} \frac{1}{6}, & i = 1, 2, \ldots, 6, \\ 0, & \text{OTHERWISE.} \end{cases}$$

$$\Rightarrow \quad F_X(c) = \begin{cases} 0 & c < 0 \\ \frac{1}{6} & 1 \leq c < 2 \\ \frac{2}{6} & 2 \leq c < 3 \\ \frac{3}{6} & 3 \leq c < 4 \\ \frac{4}{6} & 4 \leq c < 5 \\ \frac{5}{6} & 5 \leq c < 6 \\ \frac{6}{6} & 6 \leq c \end{cases}$$

DEFINE $Y := F_X^{-1}(U) = \min\{c : F_X(c) \geq U\} \quad U \in [0,1]$.

$\therefore \quad g(U) := F_X^{-1}(U)$

TO OBTAIN THE INVERSE FUNCTION :

$$F_X^{-1}(u) = \begin{cases} 1 & 0 < u \leq \frac{1}{6} \\ 2 & \frac{1}{6} < u \leq \frac{2}{6} \\ 3 & \frac{2}{6} < u \leq \frac{3}{6} \\ 4 & \frac{3}{6} < u \leq \frac{4}{6} \\ 5 & \frac{4}{6} < u \leq \frac{5}{6} \\ 6 & \frac{5}{6} < u \leq \frac{6}{6} \end{cases}$$

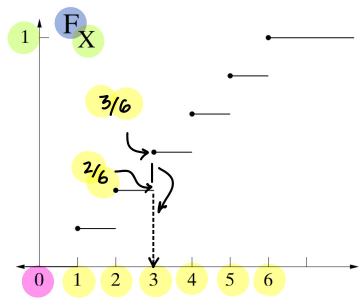


Figure 3.1: CDF for the roll of a fair die.

$$g(u) = i \quad , \quad \frac{i-1}{6} < u \leq \frac{i}{6} \qquad 1 \leq i \leq 6$$