ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 26 : ML PARAMETER ESTIMATION OF CONT-TYPE RV AND FUNCTIONS OF A RV

- TOPICS TO COVER (BASED ON CH 3.7 - 3.8)
  - → ML PARAMETER ESTIMATION OF CONT-TYPE RV
  - → FUNCTIONS OF A RV

→ ML PARAMETER ESTIMATION OF CONT-TYPE RV

$X \sim \underbrace{f_{X,\theta}}_{\text{PDF}}$ ← θ: UNKNOWN PARAMETER : CONT. CASE

GOAL : HOW TO ESTIMATE θ?

RECALL THE DISCRETE CASE :

$X \sim \underbrace{p_{X,\theta}}_{\text{PMF}}$ ← θ: UNKNOWN PARAMETER : DISCRETE CASE

$$\hat{\theta}_{MLE} := \arg\max_{\theta} \underbrace{L(\theta)}_{\text{LIKELIHOOD FUNCTION}}$$

where $L(\theta) := \underbrace{p_{X,\theta}}_{\text{PMF}}(x) = P\{X = x\}$

IN CONT. CASE :

$$f_{X,\theta}(x) \neq P\{X = x\}$$

HOWEVER, $f_{X,\theta}(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} P\{x - \frac{\varepsilon}{2} \leq X \leq x + \frac{\varepsilon}{2}\}$

$\Rightarrow$ $f_{X,\theta}(x) \propto$ PROBABILITY THAT X IS IN A SMALL NEIGHBORHOOD AROUND x

WITH CONSTANT OF PROPORTIONALITY : $\frac{1}{\varepsilon}$

$\therefore$ DEFINE $L(\theta) := f_{X,\theta}(x)$ IN CONT. CASE

$$\Rightarrow \quad \hat{\theta}_{MLE} = \underset{\theta}{\arg\max}\, L(\theta) = \underset{\theta}{\arg\max}\, f_{X,\theta}(x)$$

EXAMPLE: $X \sim$ EXPONENTIAL (λ)

$$f_{X,\lambda}(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases}$$

$$L(\lambda) = \lambda e^{-\lambda x}$$

$$\Rightarrow \quad \ell(\lambda) := \ln L(\lambda) = \ln \lambda - \lambda x$$

$$\Rightarrow \quad \ell'(\lambda) = \frac{\partial}{\partial \lambda}\ell(\lambda) = \frac{1}{\lambda} - x = 0$$

$$\Rightarrow \quad \hat{\lambda}_{MLE} = \frac{1}{x}$$

EXAMPLE: $X \sim$ UNIFORM $[0, b]$

$$f_{X,b}(x) = \begin{cases} 1/b, & 0 \leq x \leq b, \\ 0, & \text{OTHERWISE.} \end{cases}$$

$$L(b) = 1/b, \quad 0 \leq x \leq b$$

$$\ell(b) = -\ln(b)$$

$$\Rightarrow \quad \ell'(b) = \underbrace{-\frac{1}{b}}_{\leq 0} \quad (= 0 \text{ DOESN'T GIVE A MAXIMA!})$$

$\Rightarrow$ $\ell(b)$ IS A DECREASING FUNCTION OF b

$b \uparrow$ $\ell(b) \downarrow$ AND $b \downarrow$ $\ell(b) \downarrow$

SMALLEST VALUE OF b SUCH THAT $L(b) > 0$

$$L(b) = \begin{cases} 1/b, & x \le b, \\ 0, & \text{OTHERWISE.} \end{cases}$$

$$\Rightarrow \quad \hat{b}_{MLE} = x$$

→ FUNCTIONS OF A RV

$X \sim f_X$ DEFINE $Y = g(X)$ $g(\cdot)$ IS SOME FUNCTION

QUESTION: FIND THE PDF OF Y, f_Y.

RECALL $Y = aX + b$: LINEAR FUNCTION 'SCALING'

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

$g(\cdot)$ IS NOT LINEAR ALWAYS! IN GENERAL, WE HAVE A THREE-STEP PROCESS.

STEP 1. SCOPE OF THE PROBLEM:

- FIND THE SUPPORT OF X.
- SKETCH f_X AND g.
- FIND THE SUPPORT OF Y.
- DETERMINE IF Y IS CONT OR DISCRETE.

E.G. $X \sim \text{EXPONENTIAL}(1)$ $Y = g(X) := aX + b$

- SUPPORT OF X: $[0, \infty)$ AND
- SUPPORT OF Y: $[a, \infty)$

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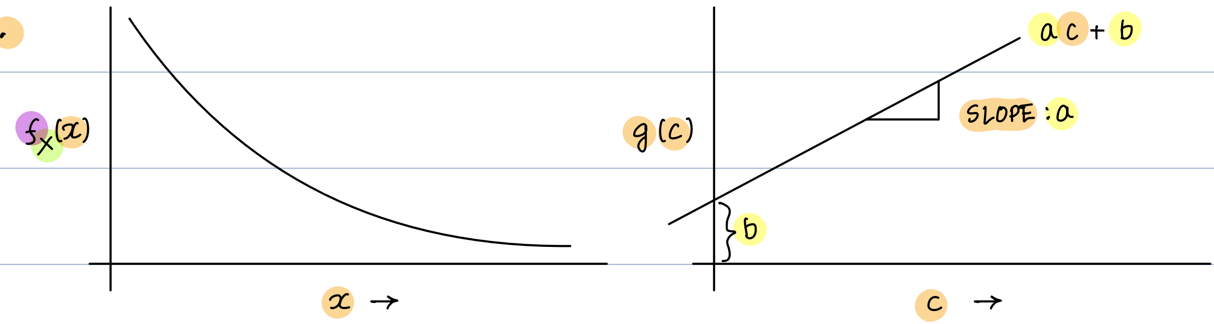

• Y IS CONTINUOUS.

IN GENERAL Y IS NOT ALWAYS CONT. E.G.

$$Y = \begin{cases} 1, & X \in [a, b], \\ 0, & \text{OTHERWISE}. \end{cases}$$

$$P\{Y=1\} = P\{X \in [a,b]\} \neq 0$$

$\Rightarrow$ Y IS DISCRETE ! $Y \sim \text{BERNOULLI}(P\{X \in [a,b]\})$

STEP 2. IF X IS CONT., FIND THE CDF OF Y

$$F_Y(c) := P\{Y \leq c\}$$
$$= P\{g(X) \leq c\}$$

STEP 3. $f_Y(x) := F_Y'(x)$

WHAT IF Y IS DISCRETE TYPE : TO BE DISCUSSED!

EXAMPLE: $X \sim f_X$

$$f_X(x) = \frac{e^{-|u|}}{2}, \quad u \in \mathbb{R}. \quad : \text{DOUBLE-EXPONENTIAL DIST.}$$

DEFINE $Y = g(X) = X^2$. WHAT IS f_Y?

STEP 1. SCOPE: SUPPORT OF X : $\mathbb{R}$

SUPPORT OF Y : $[0, \infty)$: $\mathbb{R}_+$

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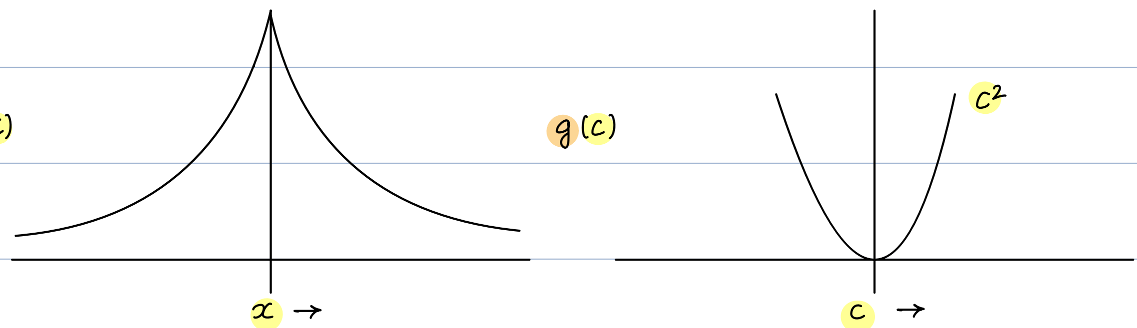

- Y IS CONTINUOUS!

STEP 2.
$$F_Y(c) := P\{Y \le c\}$$
$$= P\{X^2 \le c\}$$
$$= P\{-\sqrt{c} \le X \le \sqrt{c}\}$$

$X^2 \le c \Leftrightarrow -\sqrt{c} \le X \le \sqrt{c}$ ⇝ HOW?

$$= \int_{-\sqrt{c}}^{\sqrt{c}} f_X(u)\, du$$

$$= \int_{-\sqrt{c}}^{\sqrt{c}} e^{-\frac{|u|}{2}}\, du$$

$$= \int_{-\sqrt{c}}^{0} e^{-\frac{|u|}{2}}\, du + \int_{0}^{\sqrt{c}} e^{-\frac{|u|}{2}}\, du$$

$$= \int_{-\sqrt{c}}^{0} e^{\frac{u}{2}}\, du + \int_{0}^{\sqrt{c}} e^{-\frac{u}{2}}\, du$$

$$= 2\int_{0}^{\sqrt{c}} e^{-\frac{u}{2}}\, du$$

SOLVE!

$$= 1 - e^{-\sqrt{c}}$$

$$\Rightarrow \quad F_Y(c) = \begin{cases} 1 - e^{-\sqrt{c}}, & c \geq 0, \\ 0, & \text{OTHERWISE.} \end{cases}$$

STEP 3.

$$f_Y(c) = F_Y'(c)$$

$$= \begin{cases} e^{-\sqrt{c}}/2\sqrt{c}, & c > 0, \\ 0, & \text{OTHERWISE.} \end{cases}$$

NOTE: WE CHOSE $c > 0$ AS WHEN $c = 0$, THE FUNCTION $e^{-\sqrt{c}}/2\sqrt{c}$ IS UNDEFINED, AND CHANGING A PDF FOR SOME (COUNTABLY MANY) POINTS DOESN'T MATTER!