ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 24 : LINEAR SCALING OF PDFS AND GAUSSIAN DISTRIBUTION

- TOPICS TO COVER (BASED ON CH 3.6)
  - → LINEAR SCALING OF PDFS
  - → GAUSSIAN DISTRIBUTION

→ LINEAR SCALING OF PDFS

$X \sim f_X$ PDF OF X

DEFINE $Y = aX + b$ $\quad a > 0$

$\Rightarrow$ PDF OF Y: $\quad f_Y(v) = f_X\left(\frac{v-b}{a}\right) \cdot \frac{1}{a}$.

PROOF: CONSIDER THE CDF OF Y:

$$F_Y(v) \overset{\text{DEFINITION}}{=} P\{Y \le v\}$$

$$= P\{aX + b \le v\}$$

$$= P\left\{X \le \frac{v-b}{a}\right\} \quad \text{as } a > 0$$

$$= F_X\left(\frac{v-b}{a}\right)$$

$$\Rightarrow f_Y(v) = F_Y'(v) = \frac{d}{dv} F_X\left(\frac{v-b}{a}\right)$$

CHAIN RULE

$$= f_X\left(\frac{v-b}{a}\right) \cdot \frac{1}{a}$$

FURTHERMORE: $EY = aEX + b$ AND $\mathrm{Var}\,Y = a^2\,\mathrm{Var}\,X$

EXAMPLE: X : TEMPERATURE IN CELSIUS

Y : " IN FAHRENHEIT

$$Y = 1.8X + 32$$

$$\Rightarrow \quad f_Y(c) = f_X\left(\frac{c-32}{1.8}\right) \cdot \frac{1}{1.8}$$

IF $X \sim$ UNIFORM $[15, 20]$

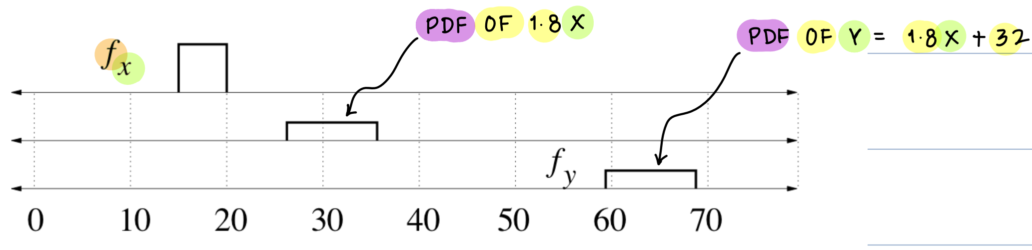
$$\Rightarrow \quad f_X(c) = \begin{cases} \frac{1}{5}, & 15 \le c \le 20, \\ 0, & \text{OTHERWISE}. \end{cases}$$

$Y = 1.8X + 32$ THEN $X \in [15, 20] \Rightarrow Y \in [\underbrace{1.8(15) + 32}_{59}, \underbrace{1.8(20) + 32}_{68}]$

$$\therefore \quad f_Y(c) = \begin{cases} \frac{1}{5} \cdot \frac{1}{1.8}, & 59 \le c \le 68, \\ 0, & \text{OTHERWISE}. \end{cases}$$

$$f_Y(c) = \begin{cases} \frac{1}{9}, & 59 \le c \le 68, \\ 0, & \text{OTHERWISE}. \end{cases}$$

$\Rightarrow \quad Y \sim$ UNIFORM $[59, 68]$



THE GAUSSIAN (NORMAL) DISTRIBUTION

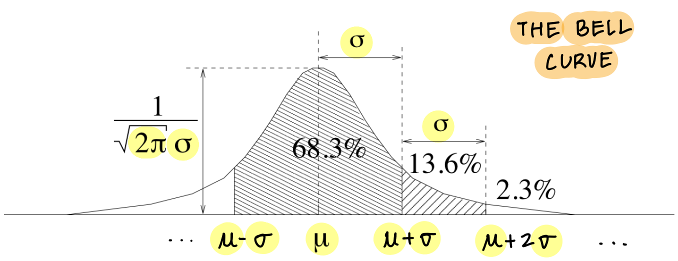
A RV Y IS SAID TO FOLLOW THE GAUSSIAN (NORMAL) DISTRIBUTION IF ITS PDF IS

$$f_Y(u) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2}, \qquad u \in \mathbb{R}, \ \underbrace{\mu \in \mathbb{R} \text{ AND } \sigma > 0}_{\text{PARAMETERS}} \qquad \cdots (*)$$

$$\sim N(\mu, \sigma^2)$$

PROPERTIES:

- $\int_{\mathbb{R}} f_Y(u)\, du = 1$
- $EY = \int_{\mathbb{R}} u\, f_Y(u)\, du = \mu$
- $\operatorname{var} Y = \int_{\mathbb{R}} u^2 f_Y(u)\, du - \mu^2 = \sigma^2$



f_Y IS SYMMETRIC ABOUT μ

Figure 3.11: The Gaussian (or normal) pdf

THE STANDARD NORMAL DISTRIBUTION

DEFINE A NEW RV X BY TAKING $\mu = 0$ AND $\sigma^2 = 1$ IN $(*)$

ALSO DENOTED AS $\phi(u)$

$$\Rightarrow f_X(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}, \qquad u \in \mathbb{R}$$

$X \sim N(0,1)$: STANDARD NORMAL DISTRIBUTION

PROPERTIES:

- $\int_{\mathbb{R}} f_X(u)\,du = 1$
- $f_X(u) = f_X(-u)$: f_X IS AN EVEN FUNCTION
- $EX = 0$
- $\operatorname{Var} X = 1$
- CDF OF $X \sim N(0,1)$:

HOW TO SOLVE THIS INTEGRAL? : TABLES

$$F_X(u) = \int_{-\infty}^{u} f_X(v)\,dv := \Phi(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv$$

- $\underbrace{\text{COMPLEMENTARY CDF}}_{\text{ALSO KNOWN AS SURVIVAL FUNCTION}}$ OF $X \sim N(0,1)$:

$$Q(u) := 1 - F_X(u) = 1 - \Phi(u) = 1 - \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv$$

$$= \int_{u}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\,dv$$

$$= \Phi(-u)$$

RELATIONSHIP BETWEEN $N(0,1)$ AND $N(\mu, \sigma^2)$

IF $X \sim N(0,1)$ THEN $Y = \sigma X + \mu \sim N(\mu, \sigma^2)$, $\sigma > 0$, $\mu \in \mathbb{R}$

PROOF:

$$f_X(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}, \quad u \in \mathbb{R}$$

USING THE SCALING RULE:

$$f_Y(u) = f_X\left(\frac{u-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2} \quad u \in \mathbb{R},\ \mu \in \mathbb{R},\ \text{AND } \sigma > 0$$

$\Rightarrow \quad Y \sim N(\mu, \sigma^2)$

EXAMPLE: $X \sim N(10, 16)$

FIND $P\{X \geq 15\}$, $P\{X \leq 5\}$, $P\{X^2 \geq 400\}$, AND $P\{X = 2\}$

- $P\{X \geq 15\} = P\left\{\frac{X-10}{4} \geq \frac{15-10}{4}\right\}$

$= P\left\{Z \geq \frac{5}{4}\right\} \qquad Z \sim N(0,1)$

$= Q\left(\frac{5}{4}\right)$

- $P\{X \leq 5\} = P\left\{\frac{X-10}{4} \leq \frac{5-10}{4}\right\}$

$= P\left\{Z \leq -\frac{5}{4}\right\} \qquad Z \sim N(0,1)$

$= \Phi\left(-\frac{5}{4}\right) = Q\left(\frac{5}{4}\right)$

- $P\{X^2 \geq 400\} = P\{X \geq 20 \text{ OR } X \leq -20\}$

$= P\{X \geq 20\} + P\{X \leq -20\}$

$= P\left\{\frac{X-10}{4} \geq \frac{20-10}{4}\right\} + P\left\{\frac{X-10}{4} \leq \frac{-20-10}{4}\right\}$

$= P\{Z \geq 2.5\} + P\{Z \leq -7.5\}$

$= Q(2.5) + \Phi(-7.5)$

$= \Phi(-2.5) + \Phi(-2.5)$

$\approx \Phi(-2.5)$ AS $\Phi(-2.5) \approx 0$

- $P\{X = 2\} = 0 \quad \forall\ X$ CONT-TYPE RV