LECTURE 23: RANDOM PROCESSES

- TOPICS TO COVER (BASED ON CH 3.5)

  → INTRODUCTION

  → BERNOULLI PROCESSES

  → POISSON PROCESSES

→ INTRODUCTION

A RANDOM PROCESS IS AN INFINITE SEQUENCE OF RANDOM VARIABLES USUALLY OCCURING OVER TIME.

→ BERNOULLI PROCESSES

$X_1, X_2, X_3, \ldots$ $\quad X_i \sim$ BERNOULLI (p) $\forall\, i$ INDEPENDENTLY

RANDOM OUTCOME OF THE i th BERNOULLI TRIAL IN AN INFINITE SEQ. OF TRIALS

E.G. SYSTEM FAILURE EVENTS OVER AN INFINITE SEQ. OF DISCRETE TIME POINTS

RECALL $(\Omega, \mathcal{F}, P)$: PROBABILITY SPACE

FOR ANY $\omega \in \Omega$: $X_k(\omega)$: REALIZED VALUE OF X_k

TIME

$X_1(\omega), X_2(\omega), X_3(\omega), \ldots, X_k(\omega), \ldots$: SAMPLE PATH OF A BERNOULLI PROCESS FOR OUTCOME $\omega \in \Omega$

REFER TO THE FOLLOWING FIGURE:

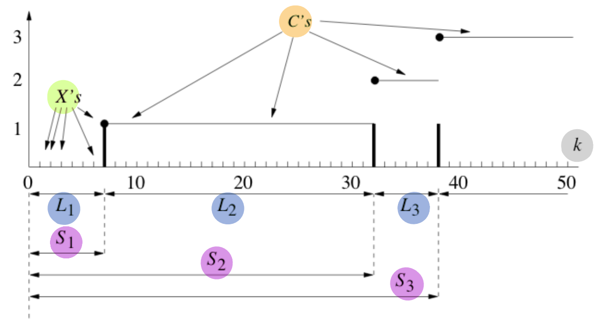

Figure 2.6: Depiction of a sample path of a Bernoulli process.

L_1 : # OF TRIALS NEEDED UNTIL THE OUTCOME OF A TRIAL IS ONE

$L_1 \sim$ GEOMETRIC (p)

L_2 : # OF TRIALS, AFTER THE FIRST L_1 TRIALS, UNTIL AGAIN THE OUTCOME OF A TRIAL IS ONE.

$L_2 \sim$ GEOMETRIC (p)

L_j : # OF TRIALS NEEDED AFTER THE FIRST $L_1 + L_2 + \cdots + L_{j-1}$ TRIALS, UNTIL THE OUTCOME OF A TRIAL IS ONE.

$L_1, L_2, \ldots, L_j, \ldots$ INDEPENDENT RVS EACH FOLLOWING GEOMETRIC (p)

NOTE THAT THE VARIABLES $L_1, L_2, \ldots, L_j, \ldots$ ARE DETERMINED BY $X_1, X_2, \ldots, X_j, \ldots$ AND VICE-VERSA!

WE HAVE TWO MORE WAYS TO DESCRIBE THE BERNOULLI RANDOM PROCESS.

→ S_j : # OF TRIALS, COUNTING FROM THE VERY FIRST ONE, UNTIL A TOTAL OF j TRIALS HAVE OUTCOME ONE

$$S_j = L_1 + L_2 + \cdots + L_j \qquad j \geq 1$$

ALSO, $L_j = S_j - S_{j-1}$ WITH $S_0 = 0$

$$S_j \sim \text{NEGATIVE BINOMIAL}(j, p)$$

→ C_k : CUMULATIVE # OF ONES IN THE FIRST k TRIALS

$$C_k = X_1 + X_2 + \cdots + X_k \qquad k \geq 1$$

ALSO, $X_k = C_k - C_{k-1}$ WITH $C_0 = 0$

$$C_k \sim \text{BINOMIAL}(k, p)$$

→ POISSON PROCESSES

RECALL THE BERNOULLI PROCESS DISCUSSED ABOVE. LET $h > 0$, WITH h REPRESENTING THE AMOUNT OF TIME. SUPPOSE EACH TRIAL IN THE BERNOULLI PROCESS TAKES h TIME UNITS TO PERFORM. A TIME-SCALED BERNOULLI RANDOM PROCESS TRACKS THE # OF COUNTS VERSUS TIME. REFER TO THE FOLLOWING FIGURE:

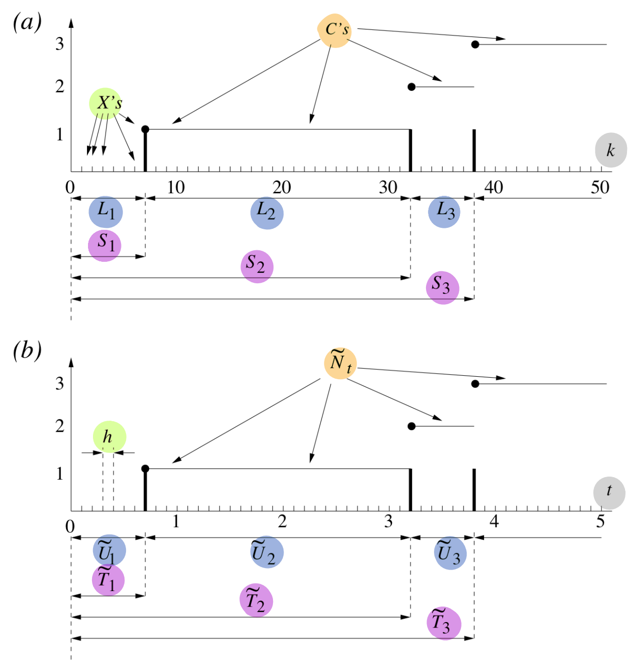

Figure 3.8: (a) A sample path of a Bernoulli process and (b) the associated time-scaled sample path of the time-scaled Bernoulli process, for $h = 0.1$.

DEFINE THE FOLLOWING RVS:

- $\widetilde{U}_j = hL_j$: AMOUNT OF TIME BETWEEN THE $(j-1)$TH COUNT AND THE jTH COUNT
- $\widetilde{T}_j = hS_j$: THE TIME THE jTH COUNT OCCURS
- $\widetilde{N}_t = C_{\lfloor t/h \rfloor}$: # OF COUNTS UPTO TIME t

TIME-SCALED BERNOULLI PROCESS

WE NEXT INTRODUCE THE POISSON PROCESS.

LET λ IS FIXED AND h IS SO SMALL THAT $p = \lambda h$ IS MUCH SMALLER THAN 1

WE CAN APPROXIMATE THE SCALED-BERNOULLI PROCESS AS FOLLOWS:

$L_j \sim \text{Geometric}(p)$
WHY? $hL_j \sim$ CONT. ANALOG OF GEOMETRIC

- $L_j \sim \text{GEOMETRIC}(p)$

AND SCALED VERSION OF L_j, NAMELY $\tilde{U}_j = h L_j \sim \text{EXPONENTIAL}(\lambda = p/h)$

- t FIXED, $\tilde{N}_t$: SUM OF $\lfloor t/h \rfloor$ BERNOULLI(p) RVS

$$\Rightarrow \quad \tilde{N}_t \sim \text{BINOMIAL}(\lfloor t/h \rfloor,\ p = \lambda h)$$

$$\therefore \quad E\tilde{N}_t = \lfloor t/h \rfloor \cdot \lambda h \approx \lambda t$$

RECALL THAT AS $n \to \infty$ AND $p \to 0$ WITH $np \to \lambda$

$$\text{BINOMIAL}(n, p) \longrightarrow \text{POISSON}(\lambda)$$

WE HAVE $h \to 0 \Rightarrow p = \lambda h \to 0$
$\lfloor t/h \rfloor \to \infty \quad \lfloor t/h \rfloor \cdot \lambda h \to \lambda t$

THINK ABOUT IT!

$$\therefore \text{ AS } h \to 0, \quad \tilde{N}_t \longrightarrow \text{POISSON}(\lambda t)$$

MORE GENERALLY, IF $0 \le s < t$, THEN

$$\tilde{N}_t - \tilde{N}_s \longrightarrow \text{POISSON}(\lambda(t-s)) \quad \text{INDEPENDENTLY}$$

FORMAL DEFINITION OF A POISSON PROCESS

GIVEN $(\Omega, \mathcal{F}, P)$ A SAMPLE PATH OF A POISSON PROCESS (i.e., THE FUNCTION OF TIME THE PROCESS YIELDS FOR SOME PARTICULAR $\omega \in \Omega$) IS SHOWIN IN THE FOLLOWING FIGURE:

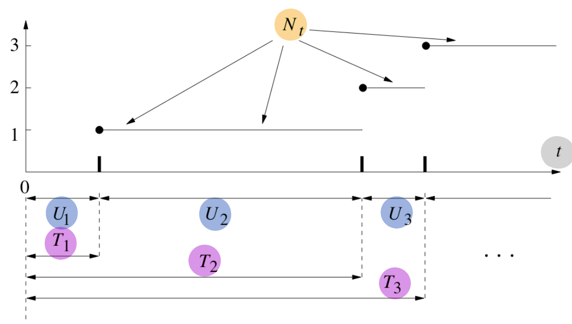

Figure 3.9: A sample path of a Poisson process.

- $t \geq 0$, N_t : CUMULATIVE # OF COUNTS UP TO TIME t
- $T_1, T_2, \ldots$: COUNT TIMES
- $U_1, U_2, \ldots$: INTER COUNT TIMES

$$N_t = \sum_{n=1}^{\infty} I_{\{t \geq T_n\}} \qquad \text{WHERE} \quad I_A = \begin{cases} 1, & A \text{ TRUE,} \\ 0, & \text{OTHERWISE.} \end{cases}$$

$$T_n = \text{MIN}\{t : N_t \geq n\}$$

$$T_n = U_1 + U_2 + \cdots + U_n$$

LET $\lambda > 0$. A POISSON PROCESS WITH RATE λ IS A RANDOM COUNTING PROCESS

$N = (N_t : t \geq 0)$ SUCH THAT

N.1 N HAS INDEP. INCREMENTS: IF $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$, THE INCREMENTS

$N_1 - N_{t_0}, N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}}$ ARE INDEPENDENT.

N.2 THE INCREMENT $N_t - N_s$ HAS THE POISSON $(\lambda(t-s))$ DISTRIBUTION FOR $t \geq s$.

PROPOSITION: LET N BE A RANDOM COUNTING PROCESS AND LET $\lambda > 0$. THE FOLLOWING ARE EQUIVALENT:

(a) N IS A POISSON PROCESS WITH RATE λ.

(b) THE INTERCOUNT TIMES $U_1, U_2, \dots,$ ARE MUTUALLY INDEP., EXPONENTIALLY DISTRIBUTED RVS WITH PARAMETER λ.