ECE 313 / MATH 362 PROBABILITY WITH ENGINEERING APPLICATIONS

LECTURE 22: CONTINUOUS-TYPE RANDOM VARIABLES AND PROBABILITY DENSITY FUNCTIONS

- TOPICS TO COVER (BASED ON CH 3.1)
  - → CONTINUOUS-TYPE RANDOM VARIABLES AND PROBABILITY DENSITY FUNCTIONS
  - → UNIFORM AND EXPONENTIAL DISTRIBUTIONS

→ CONTINUOUS-TYPE RANDOM VARIABLES AND PROBABILITY DENSITY FUNCTIONS

DEFINITION. A RV X IS A CONTINUOUS-TYPE RV IF THERE EXISTS A CONT. FUNCTION f_X, CALLED THE PROBABILITY DENSITY FUNCTION (PDF) OF X, SUCH THAT

$$P\{X \le c\} := \underbrace{F_X(c)}_{\text{CDF OF } X} = \overbrace{\int_{-\infty}^{c}}^{\text{INTEGRAL}} \overbrace{f_X(u)}^{\text{PDF}}\, du \qquad \forall\, c \in \mathbb{R} \qquad \cdots (*)$$

SUPPORT OF A PDF f_X : $\{u : f_X(u) > 0\}$

ANALOGY : FOR A DISCRETE RV X:

$$\overbrace{F_X(c)}^{P\{X \le c\}} := \overbrace{\sum_{u \le c}}^{\text{SUM}} \overbrace{p_X(u)}^{\text{PMF}} \qquad \forall\, c \in \mathbb{R}$$

USING FUNDAMENTAL THEOREM OF CALCULUS $(*)$ GIVES:

$$f_X(c) = \frac{d}{dx} \overbrace{F_X(x)}^{\text{DIFFERETIABLE}} \Big|_{x=c}$$

F_X IS DIFFERENTIABLE ⇒ F_X IS CONTINUOUS ⇒ F_X HAS NO DISCONTINUITIES

⇒ F HAS NO JUMPS

RIGHT LIMIT OF F AT x — LEFT LIMIT OF F AT x — F AT x

$$\Rightarrow \quad F_X(x+) = F_X(x-) = F_X(x) \qquad \cdots (1)$$

OBSERVATION 1.

$$P\{X = a\} = P\{X \le a\} - P\{X < a\}$$

DEF. — PROPERTY WE SAW

$$= F_X(a) - F_X(a-)$$

(1)

$$= 0 \qquad \cdots (2)$$

OBSERVATION 2.

$a < b$;

$$P\{a < X \le b\} = P\{X \le b\} - P\{X \le a\}$$

$$P\{X \in (a, b]\} = F_X(b) - F_X(a)$$

$$= \int_{-\infty}^{b} f_X(u)\,du - \int_{-\infty}^{a} f_X(u)\,du$$

$$= \int_{a}^{b} f_X(u)\,du \qquad \cdots (3)$$

AS $P\{X = c\} = 0 \;\; \forall \, c \in \mathbb{R}$ $\Rightarrow$

$$P\{X \in (a, b)\} = P\{X \in [a, b]\}$$

$$= P\{X \in (a, b]\}$$

$$= P\{X \in [a, b)\}$$

ANOTHER WAY OF MAKING OBSERVATION 1:

$$P\{X = a\} = P\{a < X \le a\}$$

USING (3)

$$= \int_{a}^{a} f_X(u)\,du = 0 \quad \text{(ALWAYS!)}$$

OBSERVATION 3.

$$P\{X \in \mathbb{R}\} = P\{X \in (-\infty, \infty)\}$$

1 (ALWAYS!)

$$= \lim_{a \to -\infty} \lim_{b \to \infty} P\{X \in (a, b]\}$$

$$= \lim_{a \to -\infty} \lim_{b \to \infty} \int_a^b f_X(u)\, du$$

$$1 = \int_{-\infty}^{\infty} f_X(u)\, du \quad \cdots (4)$$

THE PDF INTEGRATES TO 1 OVER THE REAL-LINE !

MEAN AND VARIANCE OF A CONT-TYPE RV :

- MEAN $\mu_X = EX = \displaystyle\int_{-\infty}^{\infty} u\, f_X(u)\, du$

- VARIANCE $\sigma_X^2 = \mathrm{Var}\, X = EX^2 - (EX)^2 = \displaystyle\int_{-\infty}^{\infty} u^2 f_X(u)\, du - \left(\int_{-\infty}^{\infty} u\, f_X(u)\, du \right)^2$

- LOTUS $E(g(X)) = \displaystyle\int_{-\infty}^{\infty} g(u)\, f_X(u)\, du$

- $E(aX^2 + bX + c) \overset{\text{LINEARITY}}{=} a\,E(X^2) + b\,E(X) + c$

- $\mathrm{Var}(aX + c) = a^2\, \mathrm{Var}(X)$

- STANDARDIZED RV : $Z = \dfrac{X - \mu}{\sigma}$;

$\mu_Z = 0$ AND

$\sigma_Z^2 = 1$

EXAMPLE : X IS A CONT-TYPE RV WITH THE FOLLOWING PDF

CONSTANT (UNKNOWN)

$$f_X(u) = \begin{cases} A(1-u^2), & u \in [-1, 1], \\ 0, & \text{Otherwise}. \end{cases}$$

FIND A, $P\{0.5 < X < 1.5\}$, F_X, μ_X, AND σ_X^2.

- FINDING A : WE KNOW THE FOLLOWING:

$$\int_{-\infty}^{\infty} f_X(u)\, du = 1$$

$$\Rightarrow \int_{-1}^{1} A(1-u^2)\, du + \int_{u \notin (-1,1)} 0\, du = 1$$

$$\Rightarrow A\left(u - \frac{u^3}{3}\right)\Bigg|_{-1}^{1} + 0 = 1$$

$$\Rightarrow \frac{4A}{3} = 1$$

$$\Rightarrow A = \frac{3}{4}$$

- $$P\{.5 < X < 1.5\} = \int_{.5}^{1.5} f_X(u)\, du$$

$$= \int_{.5}^{1} \frac{3}{4}(1-u^2)\, du + 0$$

$$= \frac{3}{4}\left(u - \frac{u^3}{3}\right)\Bigg|_{.5}^{1}$$

$$= \frac{3}{4}\left(\frac{2}{3} - \frac{11}{24}\right)$$

$$= \frac{5}{32}$$

• FINDING F_X :

$$F_X(c) = \int_{-\infty}^{c} f_X(u)\, du \qquad c \in \mathbb{R}$$

$\forall \quad c \le -1 \quad : \quad F_X(c) = \int_{-\infty}^{c} 0 \cdot du = 0$

$\forall \quad c \in (-1, 1) \quad : \quad F_X(c) = \int_{-\infty}^{-1} 0\, du + \int_{-1}^{c} \frac{3}{4}(1-u^2)\, du$

$$= \frac{3}{4}\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{c}$$

$$= \frac{2 + 3c - c^3}{4}$$

$\forall \quad c \ge 1 \quad : \quad F_X(c) = \int_{-\infty}^{-1} 0\, du + \int_{-1}^{1} \frac{3}{4}(1-u^2)\, du + \int_{1}^{c} 0\, du$

$$= 0 + 1 + 0 = 1$$

$$\Rightarrow \quad F_X(c) = \begin{cases} 0, & c \le -1, \\ \frac{2+3c-c^3}{4}, & c \in (-1, 1) \\ 1, & c \ge 1 \end{cases}$$

F IS CONT. $\Rightarrow$ F HAS NO JUMPS!

CAN YOU SEE THIS?

• $EX = \int_{-\infty}^{\infty} u\, f_X(u)\, du = 0$

EXERCISE!

• $\operatorname{var} X = EX^2 - (EX)^2 = EX^2 = \int_{-\infty}^{\infty} u^2 f_X(u)\, du = 0.2$

0

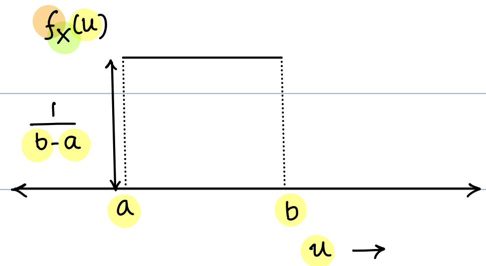
• UNIFORM DISTRIBUTION :

$a < b$; $X \sim$ UNIFORM $[a, b]$ IF

PDF OF X

SUPPORT OF f_X

$$f_X(u) = \begin{cases} \dfrac{1}{b-a}, & u \in [a, b], \\ 0, & \text{otherwise}. \end{cases}$$



→ CDF OF X :

$$F_X(c) = P\{X \le c\} = \int_{-\infty}^{c} f_X(u)\, du$$

• IF $c \le a$: $F_X(c) = \int_{-\infty}^{c} f_X(u)\, du = \int_{-\infty}^{c} 0\, du = 0$

• IF $c \in (a, b)$: $F_X(c) = \int_{-\infty}^{c} f_X(u)\, du = \int_{-\infty}^{a} 0\, du + \int_{a}^{c} \frac{1}{b-a}\, du$

$$= \frac{c}{b-a}$$

• IF $c \ge b$: $F_X(c) = \int_{-\infty}^{c} f_X(u)\, du$

$$= \int_{-\infty}^{a} 0\, du + \int_{a}^{b} \frac{1}{b-a}\, du + \int_{b}^{\infty} 0\, du$$

$$= \frac{b-a}{b-a} = 1$$

→ E X

$$E X = \int_{-\infty}^{\infty} u\, f_X(u)\, du = \int_{a}^{b} u \frac{1}{b-a}\, du = \frac{1}{b-a} \int_{a}^{b} u\, du$$

$$= \frac{1}{b-a} \cdot \frac{u^2}{2} \Big|_{a}^{b} = \frac{1}{b-a} \frac{(b-a)^2}{2}$$

$$= \frac{(b-a)(b+a)}{(b-a)\; 2} = \frac{b+a}{2}$$

$\rightarrow \quad EX^2 = \dfrac{a^2 + ab + b^2}{3}$ EXERCISE!

$\Rightarrow \quad \mathrm{Var}\, X = \dfrac{(b-a)^2}{12}$

NOTE: 'STANDARD' UNIFORM DIST. : IF $a = 0$ AND $b = 1$, i.e., $Z \sim$ UNIFORM $[0, 1]$

$$f_Z(u) = \begin{cases} 1, & u \in [0, 1], \\ 0, & \text{otherwise}. \end{cases}$$

$$F_Z(c) = \begin{cases} 0, & c \leq 0, \\ c, & c \in (0, 1), \\ 1, & c \geq 1 \end{cases}$$

$$EZ = \frac{1}{2}$$

$$\mathrm{Var}\, Z = \frac{1}{12}$$

- EXPONENTIAL DISTRIBUTION

$\lambda > 0$; $\quad T \sim$ EXPONENTIAL (λ) IF

PDF

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0, \\ 0, & \text{otherwise}. \end{cases}$$

CDF OF X :

$$F_T(t) = 0 \quad \text{if } t \leq 0$$

$$F_T(t) = \int_0^t f_T(s)\, ds \quad t \geq 0$$

$$= \int_0^t \lambda e^{-\lambda s}\, ds$$

$$= \lambda \cdot \left. \frac{e^{-\lambda s}}{-\lambda} \right|_0^t$$

$$= 1 - e^{-\lambda t} \quad t \geq 0$$

$$\therefore \quad F_T(t) = \begin{cases} 1 - e^{-\lambda t}, & t \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

nth MOMENT OF X :

$$E T^n = \int_0^\infty t^n \underbrace{\lambda e^{-\lambda t}}_{f_X(t)}\, dt$$

integration by parts

$$= -t^n e^{-\lambda t} \Big|_0^\infty + \int_0^\infty n t^{n-1} e^{-\lambda t}\, dt$$

$$= 0 + \frac{n}{\lambda} \underbrace{\int_0^\infty t^{n-1} \underbrace{\lambda e^{-\lambda t}}_{f_X(t)}\, dt}_{E T^{n-1}}$$

$$\Rightarrow \quad E T^n = \frac{n}{\lambda} E T^{n-1} \; ; \quad n \geq 1 \quad : \text{RECURRENCE RELATION}$$

$$\Rightarrow \quad E T = \frac{1}{\lambda}$$

$$\operatorname{Var} T = E T^2 - (E T)^2 = \frac{2}{\lambda} E T - (E T)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

MEMORYLESS PROPERTY OF EXPONENTIAL DIST.

$\lambda > 0$; $T \sim$ EXPONENTIAL(λ)

$$P(T > s+t \mid T > s) = \frac{P\{T > s+t,\ T > s\}}{P\{T > s\}}$$

$$= \frac{P\{T > s+t\}}{P\{T > s\}}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= e^{-\lambda t} = P\{T > t\}$$

$$F_T(u) = P\{T < u\} = 1 - e^{-\lambda u}$$

$$\Rightarrow P\{T > u\} = e^{-\lambda u}$$