**University of Illinois** **Fall 2025**

# ECE 313: Problem Set 13: Problems and Solutions

**Due:** Monday, December 8 at 7:00:00 p.m.

**Reading:** *ECE 313 Course Notes,* Sections 4.9.2, 4.9.3, 4.10.1, 4.10.2, and 4.11

**Note on reading:** For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

**Note on turning in homework:** Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. **[Covariance and Correlation]**
Random variables $X_1$ and $X_2$ represent two observations of a signal corrupted by noise. They have the same mean $\mu$ and variance $\sigma^2$. The signal-to-noise ratio (SNR) of the observation $X_1$ or $X_2$ is defined as the ratio $\mathrm{SNR}_X = \frac{\mu^2}{\sigma^2}$. A system designer chooses the averaging strategy, whereby she constructs a new random variable $S = \frac{X_1+X_2}{2}$.

   (a) Show that the SNR of $S$ is twice that of the individual observations, if $X_1$ and $X_2$ are uncorrelated.
   **Solution:**

$$\begin{aligned}
\mu_S =& \mathbb{E}[S] = \mathbb{E}\left[\frac{X_1+X_2}{2}\right] = \mu, \\
\sigma_S^2 =& \mathrm{Var}\left(\frac{X_1+X_2}{2}\right) \\
=& \frac{1}{4}\mathrm{Var}(X_1+X_2) = \frac{1}{4}\left\{\mathrm{Var}(X_1)+\mathrm{Var}(X_2)+2\,\mathrm{Cov}(X_1,X_2)\right\} \\
\overset{(*)}{=}& \frac{1}{4}\left\{\mathrm{Var}(X_1)+\mathrm{Var}(X_2)\right\}, \\
\therefore \quad \mathrm{SNR}_S =& \frac{\mu_S^2}{\sigma_S^2} = \frac{4\mu^2}{\mathrm{Var}(X_1)+\mathrm{Var}(X_2)} = \frac{2\mu^2}{\sigma^2} = 2\mathrm{SNR}_X,
\end{aligned}$$

   where $(*)$ follows from the uncorrelatedness of $X$ and $Y$. From this, one can conclude that averaging improves the SNR by a factor equal to the number of observations being averaged, if the observations are uncorrelated.

   (b) The system designer notices that the averaging strategy is giving $\mathrm{SNR}_S = (1.5)\mathrm{SNR}_X$.

She correctly assumes that the observations $X_1$ and $X_2$ are correlated. Determine the value of the correlation coefficient $\rho_{X_1,X_2}$.

**Solution:** Since $\text{Cov}(X_1, X_2) = \sigma^2 \rho_{X_1,X_2}$, the aforementioned derivation in (a) tells us that

$$\text{SNR}_S = \frac{\mu_S^2}{\sigma_S^2} = \frac{4\mu^2}{\text{Var}(X_1) + \text{Var}(X_2) + 2\,\text{Cov}(X, Y)} = \frac{2\mu^2}{\sigma^2(1 + \rho_{X_1,X_2})}.$$

Setting $\text{SNR}_S = 1.5\frac{\mu^2}{\sigma^2}$, we get $\rho_{X_1,X_2} = \frac{1}{3}$.

(c) Under what condition on $\rho_{X_1,X_2}$ can the averaging strategy result in an $\text{SNR}_S$ that is as high as possible?

**Solution:** One can observe that $\text{SNR}_S \to \infty$ as $\rho_{X_1,X_2} \to -1$.

2. **[MMSE Estimation]**

Consider the joint pdf below describing the dependence between random variables $X$ and $Y$:

$$f_{X,Y} = \begin{cases} u + v, & (u, v) \in [0, 1]^2; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

We wish to design various types of minimum mean squared error (MMSE) estimators of $Y$.

(a) Determine the MMSE optimal constant estimator $c^*$ of $Y$ and the resulting MSE.

**Solution:**

The MMSE optimal constant estimator of $Y$ is $c^* = E[Y]$ and the resulting MSE is $\text{Var}(Y)$. To find $E[Y]$, determine the marginal $f_Y(v)$ first:

$$f_Y(v) = \int_0^1 f_{X,Y}(u, v)\, du = \int_0^1 (u + v)\, du = \frac{1}{2} + v, \qquad 0 \le v \le 1. \tag{2}$$

Then we compute

$$E[Y] = \int_0^1 v f_Y(v)\, dv = \int_0^1 v \left(\frac{1}{2} + v\right) dv = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}. \tag{3}$$

Next,

$$E[Y^2] = \int_0^1 v^2 f_Y(v)\, dv = \int_0^1 v^2 \left(\frac{1}{2} + v\right) dv = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}. \tag{4}$$

Thus, the resulting MSE is

$$\text{Var}(Y) = E[Y^2] - (E[Y])^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}. \tag{5}$$

Thus, if we are restricted to using one constant scalar value to estimate $Y$ while minimizing the MSE, it would be $c^* = \frac{7}{12}$ and the minimum MSE value would be $\frac{11}{144}$.

(b) Determine the MMSE optimal unconstrained estimator $g^*(X)$ and the resulting MSE. Compare the MSE values obtained in parts (a) and (b). *Hint: you can use a numerical solver to evaluate the integral that appears in the MSE.*

**Solution:**

The MMSE optimal unconstrained estimator $g^*(X) = E[Y|X]$ and its MSE is $E[Y^2] - E[(E[Y|X])^2]$. We need to find $E[Y|X]$ first. This requires the conditional pdf $f_{Y|X}(v|u)$ to be found as follows:

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)}, \quad f_X(u) > 0. \tag{6}$$

We determine the marginal $f_X(u)$ as follows:

$$f_X(u) = \int_0^1 f_{X,Y}(u,v)\,dv = \int_0^1 (u+v)\,dv = u + \frac{1}{2}, \qquad 0 \le u \le 1. \tag{7}$$

Substituting the expression for $f_X(u)$ in (6), we get:

$$f_{Y|X}(v|u) = \frac{u+v}{u+1/2}, \qquad 0 \le u \le 1,\ 0 \le v \le 1.$$

Next, we compute $E[Y|X=u]$ as follows:

$$E[Y|X=u] = \int_0^1 v f_{Y|X}(v|u)\,dv = \frac{1}{u+1/2}\int_0^1 (uv+v^2)\,dv = \frac{\frac{u}{2}+\frac{1}{3}}{u+1/2} = \frac{3u+2}{3(2u+1)}.$$

Hence, the optimal unconstrained estimator $g^*(X) = \dfrac{3X+2}{3(2X+1)}$. The resulting MSE is calculated as follows:

$$\begin{aligned}
\text{MSE} &= E[Y^2] - E[(E[Y|X])^2] \\
&= \frac{5}{12} - E\left[\left(\frac{3X+2}{3(2X+1)}\right)^2\right] \\
&\overset{\text{LOTUS}}{=} \frac{5}{12} - \int_0^1 \left(\frac{3u+2}{3(2u+1)}\right)^2 \left(u+\tfrac{1}{2}\right) du \qquad (\text{using } f_X(u) = u + \tfrac{1}{2}) \\
&\approx 0.0757 \qquad (\text{integral evaluated numerically})
\end{aligned}$$

so numerically

$$\text{MSE}_{\text{unconstrained}} \approx 0.0757 < 0.0764 \approx \text{MSE}_{\text{const}}.$$

Comparing the MSE from Part (a) to the one in Part (b), we find that the optimal unconstrained estimator gives a smaller MSE than an optimal constant estimator as it should.

(c) Determine the MMSE linear estimator $L^*(X)$ and the resulting MSE.

**Solution:** The optimal linear estimator $L^*(X) = \mu_Y + \frac{\text{Cov}(Y,X)}{\text{Var}(X)}(X - \mu_X)$ and the resulting MSE is $\sigma_Y^2 - \frac{(\text{Cov}(Y,X))^2}{\text{Var}(X)}$.

From Part (a), we have $\mu_Y = \frac{7}{12}$ and from symmetry $\mu_X = \frac{7}{12}$. We also have $\text{Var}(X) = \text{Var}(Y) = \frac{11}{144}$.

To obtain $L^*(X)$ and its MSE, we need $\text{Cov}(Y,X)$. Since $\text{Cov}(Y,X) = E[XY] - E[X]E[Y]$ and we have the joint pdf of $X$ and $Y$, we can obtain $E[XY]$ as follows:

$$E[XY] = \int_0^1\int_0^1 uv(u+v)\,dv\,du = \frac{1}{3}.$$

Thus,

$$\text{Cov}(X, Y) = \frac{1}{3} - \left(\frac{7}{12}\right)^2 = -\frac{1}{144}.$$

Therefore, the optimal linear estimator is

$$L^*(X) = \frac{7}{12} + \frac{-1/144}{11/144}\left(X - \frac{7}{12}\right) = \frac{7}{11} - \frac{X}{11}.$$

The resulting MSE is

$$\text{MSE} = \frac{11}{144} - \frac{(1/144)^2}{11/144} = \frac{5}{66}.$$

Since the set of all possible linear estimators is a subset of the set of all possible unconstrained estimators, and since the optimal (the best) unconstrained estimator provides the smallest possible MSE, the optimal linear estimator has a larger MSE than the optimal unconstrained estimator.

3. **[MMSE Estimation]**
Let $Y \sim \text{Exp}(\lambda = 2)$ and $Z \sim \text{Exp}(\lambda = 1)$ be independent random variables. Let the observation be $X = Y + Z$.

   (a) Determine the MMSE optimal constant estimator $c^*$ of $Y$ and the resulting MSE.
   **Solution:** For a constant estimator, the MMSE-optimal choice is

   $$c^* = \mathbb{E}[Y].$$

   Since $Y \sim \text{Exp}(\lambda = 2)$, we have

   $$c^* = \mathbb{E}[Y] = \frac{1}{\lambda} = \frac{1}{2}.$$

   The resulting MSE is

   $$\text{MSE}(c^*) = \mathbb{E}\big[(Y - c^*)^2\big] = \text{Var}(Y) = \frac{1}{\lambda^2} = \frac{1}{2^2} = \frac{1}{4}.$$

   (b) Determine the MMSE-optimal (unconstrained) estimator $g^*(X)$. *(Note: You do not need to compute the resulting MSE; it is computable but involves a more complicated integral.)*
   **Solution:** Because $Y$ and $Z$ are independent, we can compute the joint pdf of $X$ and $Y$ with the product of $f_Y$ and $f_Z$. Note that $\begin{pmatrix} X \\ Y \end{pmatrix} = A \begin{pmatrix} Y \\ Z \end{pmatrix}$, where $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ with $det(A) = -1$

   $$f_{X,Y}(u, v) = \frac{1}{|det(A)|} f_{Y,Z}(v, u - v) = \begin{cases} 2e^{-2v}e^{-(u-v)} & \text{if } 0 \le v \le u \\ 0 & \text{else} \end{cases}$$

   Next, we compute the marginal pdf $f_X(u)$ and conditional pdf $f_{Y|X}(v|u)$ via

   $$f_X(u) = \int_0^u 2e^{-2v}e^{-(u-v)}dv = 2e^{-u}(1 - e^{-u})$$

   $$f_{Y|X}(v|u) = \frac{f_{X,Y}(u, v)}{f_X(u)} = \frac{e^{-v}}{1 - e^{-u}} \text{ for } 0 \le v \le u$$

In other words, the conditional distribution $f_{Y|X}(v|u)$ is a exponential distribution of $\lambda = 1$, but truncated to the interval $[0, u]$.
The unconstriant estimator

$$g^*(X) = E[Y|X = u] = \int_0^u \frac{ve^{-v}}{1 - e^{-u}} dv = \frac{1}{1 - e^{-u}} \int_0^u ve^{-v} dv$$
$$= \frac{1}{1 - e^{-u}} \left[ -(v + 1)e^{-v} \right]_0^u = \frac{1 - (u + 1)e^{-u}}{1 - e^{-u}}$$

(c) Determine the MMSE linear estimator $L^*(X)$ and the resulting MSE.
**Solution:** $L^*(X) = \mu_Y + \frac{Cov(Y,X)}{Var(X)}(X - \mu_X)$ and the corresponding MSE is $\sigma_Y^2 - \frac{(Cov(Y,X))^2}{Var(X)}$ In part (a), we have $\mu_Y = \frac{1}{2}$. Since $Y$ and $Z$ are independent, using LOTUS we can get $\mu_X = \mu_Y + \mu_Z = \frac{1}{2} + 1 = \frac{3}{2}$.
The covariance and variance can be computed via

$$Cov(X, Y) = Cov(Y + Z, Y) = Var(Y) + Cov(Z, Y) = Var(Y) = \frac{1}{4}$$
$$Var(X) = Cov(Y + Z, Y + Z) = Var(Y) + Var(Z) = \frac{1}{4} + 1 = \frac{5}{4}$$

Therefore, the optimal linear estimator is

$$L^*(X) = \frac{1}{2} + \frac{\frac{1}{4}}{\frac{5}{4}}(X - \frac{3}{2}) = \frac{1}{5}X + \frac{1}{5}$$

The resulting MSE is

$$MSE = \frac{1}{4} - \frac{(\frac{1}{4})^2}{\frac{5}{4}} = \frac{1}{5}$$

4. **[Gaussian random variables]**
Let $X$ and $Y$ be independent random variables with $X, Y \sim \mathcal{N}(0, 1)$.

(a) Find $\text{Cov}(3X - Y,\ X + 4Y + 2)$.
**Solution:** Since $X$ and $Y$ are independent with $\text{Var}(X) = \text{Var}(Y) = 1$ and $\text{Cov}(X, Y) = 0$, we can use bilinearity of covariance:

$$\begin{aligned}\text{Cov}(3X - Y,\ X + 4Y + 2) &= 3 \cdot 1 \cdot \text{Var}(X) + (-1) \cdot 4 \cdot \text{Var}(Y) + (3 \cdot 4 + (-1) \cdot 1)\,\text{Cov}(X, Y) \\ &= 3 \cdot 1 + (-4) \cdot 1 + (12 - 1) \cdot 0 \\ &= 3 - 4 \\ &= -1.\end{aligned}$$

(b) Express $\mathbb{P}(3X + Y \geq 2)$ in terms of the $Q$-function.
**Solution:** First, note that $3X + Y$ is Gaussian since it is a linear combination of independent Gaussians. We compute its mean and variance:

$$\begin{aligned}\mathbb{E}[3X + Y] &= 3\mathbb{E}[X] + \mathbb{E}[Y] = 0, \\ \text{Var}(3X + Y) &= 3^2\,\text{Var}(X) + 1^2\,\text{Var}(Y) + 2 \cdot 3 \cdot 1 \cdot \text{Cov}(X, Y) \\ &= 9 \cdot 1 + 1 \cdot 1 + 0 = 10.\end{aligned}$$

Thus $3X + Y \sim \mathcal{N}(0, 10)$, and

$$\begin{aligned}\mathbb{P}(3X + Y \geq 2) &= \mathbb{P}\left(\frac{3X + Y}{\sqrt{10}} \geq \frac{2}{\sqrt{10}}\right) \\ &= Q\left(\frac{2}{\sqrt{10}}\right).\end{aligned}$$

(c) Express $\mathbb{P}\big((2X + Y)^2 > 9\big)$ in terms of the $Q$-function.
**Solution:** Again, $2X + Y$ is Gaussian:

$$\begin{aligned}\mathbb{E}[2X + Y] &= 0, \\ \operatorname{Var}(2X + Y) &= 2^2 \operatorname{Var}(X) + 1^2 \operatorname{Var}(Y) + 2 \cdot 2 \cdot 1 \cdot \operatorname{Cov}(X, Y) \\ &= 4 + 1 + 0 = 5.\end{aligned}$$

So $2X + Y \sim \mathcal{N}(0, 5)$. Then

$$(2X + Y)^2 > 9 \iff |2X + Y| > 3.$$

By symmetry of a zero-mean Gaussian,

$$\begin{aligned}\mathbb{P}\big((2X + Y)^2 > 9\big) &= \mathbb{P}(|2X + Y| > 3) \\ &= 2\,\mathbb{P}(2X + Y > 3) \\ &= 2\,\mathbb{P}\left(\frac{2X + Y}{\sqrt{5}} > \frac{3}{\sqrt{5}}\right) \\ &= 2Q\left(\frac{3}{\sqrt{5}}\right).\end{aligned}$$

5. **[Gaussian Random variables and Estimation]**
Suppose that $X$ and $Y$ are jointly Gaussian random variables with

$$\mathbb{E}[X] = 1, \quad \mathbb{E}[Y] = -2, \quad \operatorname{Var}(X) = 4, \quad \operatorname{Var}(Y) = 9,$$

and correlation coefficient $\rho = \frac{1}{3}$. Let $W = 3X + Y - 2$.

(a) Find $\mathbb{E}[W]$ and $\operatorname{Var}(W)$.
**Solution:** We first compute the covariance $\operatorname{Cov}(X, Y)$. Since

$$\operatorname{Cov}(X, Y) = \rho\,\sigma_X \sigma_Y,$$

with $\sigma_X = \sqrt{\operatorname{Var}(X)} = 2$ and $\sigma_Y = \sqrt{\operatorname{Var}(Y)} = 3$, we obtain

$$\operatorname{Cov}(X, Y) = \frac{1}{3} \cdot 2 \cdot 3 = 2.$$

Now compute $\mathbb{E}[W]$:

$$\begin{aligned}\mathbb{E}[W] &= \mathbb{E}[3X + Y - 2] \\ &= 3\mathbb{E}[X] + \mathbb{E}[Y] - 2 \\ &= 3 \cdot 1 + (-2) - 2 = -1.\end{aligned}$$

For the variance,

$$\begin{aligned}
\operatorname{Var}(W) &= \operatorname{Var}(3X+Y-2) = \operatorname{Var}(3X+Y) \\
&= 3^2\operatorname{Var}(X) + 1^2\operatorname{Var}(Y) + 2\cdot 3\cdot 1\cdot \operatorname{Cov}(X,Y) \\
&= 9\cdot 4 + 1\cdot 9 + 2\cdot 3\cdot 2 \\
&= 36+9+12 = 57.
\end{aligned}$$

So,

$$\mathbb{E}[W] = -1, \qquad \operatorname{Var}(W) = 57.$$

(b) Find the best unconstrained estimator $g^\star(W)$ of $Y$ based on $W$ and the resulting MSE.

**Solution:** Because $(X,Y)$ are jointly Gaussian and $W$ is a linear combination of them, $(W,Y)$ are jointly Gaussian. For jointly Gaussian variables, the MMSE (best mean-square) estimator of $Y$ given $W$ is linear and given by

$$g^\star(w_0) = \mathbb{E}[Y] + \frac{\operatorname{Cov}(Y,W)}{\operatorname{Var}(W)}\big(w_0 - \mathbb{E}[W]\big).$$

First, compute $\operatorname{Cov}(Y,W)$:

$$\begin{aligned}
\operatorname{Cov}(Y,W) &= \operatorname{Cov}(Y, 3X+Y-2) \\
&= 3\operatorname{Cov}(Y,X) + \operatorname{Cov}(Y,Y) \\
&= 3\operatorname{Cov}(X,Y) + \operatorname{Var}(Y) \\
&= 3\cdot 2 + 9 = 15.
\end{aligned}$$

Thus

$$\frac{\operatorname{Cov}(Y,W)}{\operatorname{Var}(W)} = \frac{15}{57} = \frac{5}{19}.$$

Recall $\mathbb{E}[Y] = -2$ and $\mathbb{E}[W] = -1$, so

$$g^\star(w_0) = -2 + \frac{5}{19}(w_0+1).$$

This is the best unconstrained estimator of $Y$ based on $W$.

For the resulting MSE, we can use the identity

$$\text{MSE} = \mathbb{E}\big[(Y - g^\star(W))^2\big] = \operatorname{Var}(Y) - \frac{\operatorname{Cov}(Y,W)^2}{\operatorname{Var}(W)}.$$

So

$$\begin{aligned}
\text{MSE} &= \operatorname{Var}(Y) - \frac{\operatorname{Cov}(Y,W)^2}{\operatorname{Var}(W)} \\
&= 9 - \frac{15^2}{57} \\
&= \frac{96}{19}.
\end{aligned}$$

Therefore,

$$g^\star(w_0) = -2 + \frac{5}{19}(w_0+1), \quad \mathbb{E}\big[(Y - g^\star(W))^2\big] = \frac{96}{19}.$$