University of Illinois **Fall 2025**

ECE 313: Problem Set 8: Problems and Solutions

Due: Friday, Oct 31 at 11:59:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.5 - 3.6.2

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Mondays, and is due by 11:59 p.m. on the following Monday. You must upload handwritten homework to Gradescope. No typeset homework will be accepted. No late homework will be accepted. Please write on the top right corner of the first page:

NAME AS IT APPEARS ON Gradescope
NETID
SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings.

1. [**Customers in a coffee shop**]
   Alice is working in a coffee shop. She found that Customer arrivals follow a Poisson process with rate $\lambda = 4$ customers/hour.

   (a) What is the probability that no customer arrives within the next 30 minutes?
   Solution: Let X be the number of arrivals in $t = 0.5$ hours. $X \sim Pois(\lambda = 0.5 \times 4 = 2)$. The probability that no customer comes is $p_X(0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-2}$.

   (b) Alice wants to take a 30-minute nap during work. She will be reported if any customer arrives during the nap. When should Alice start her 30 mins nap to minimize the chance of being caught? Assume there's no customer at the shop right now at $T = 0$.
   Solution: In a Poisson process, $P\{\text{No arrival in the next 30mins}\} = e^{-\lambda \times 0.5} = e^{-2}$ Therefore, all start times are equally good; the probability of being reported is the same no matter when she starts.

   (c) If Alice wants the probability of being reported to be less than 50%, how long can she sleep?
   Solution: Let s be the length of the nap in hours. The customer visits during the nap follows $X \sim Pois(\lambda = 4s)$ The chance of being report is $1 - p_X(0)$ must be smaller than 0.5 We have $p_X(0) = e^{-4s} > 0.5$, $s < \frac{ln2}{4} \approx 0.173$ hours, or 10.4 minutes.

2. [**Poisson Process Intervals**]
   Let N_t be a Poisson process with rate $\lambda > 0$. Your answers may include λ for the following questions.

   (a) Find $P(N_5 = 7)$.
   Solution: Since $N_t \sim \text{Poi}(\lambda t)$,

$$P(N_5 = 7) = e^{-5\lambda}\frac{(5\lambda)^7}{7!}.$$

(b) Find $P(N_8 - N_3 = 7)$ and $E[N_8 - N_3]$.

Solution: Since $N_8 - N_3 \sim \text{Poi}(5\lambda)$, therefore

$$P(N_8 - N_3 = 7) = e^{-5\lambda}\frac{(5\lambda)^7}{7!}$$

The expected value of a $\text{Poi}(\lambda t)$ random variable is λt, therefore,

$$E[N_8 - N_3] = 5\lambda.$$

(c) Find $P(N_8 - N_3 = 7 | N_5 - N_4 = 5)$.

Solution: If we can break this problem into non-overlapping intervals, we can take advantage of their statistical independence. To this end, we can think of the properties of $N_8 - N_5$, $N_5 - N_4$, and $N_4 - N_3$. We know that $N_5 - N_4 = 5$, i.e., there were 5 counts in the time interval $[4, 5]$. Since a count of 7 is observed in the time interval $[3, 8]$, this can occur in 3 disjoint ways:

i. $N_8 - N_5 = 0$, $N_4 - N_3 = 2$

ii. $N_8 - N_5 = 1$, $N_4 - N_3 = 1$

iii. $N_8 - N_5 = 2$, $N_4 - N_3 = 0$

Therefore, the probability of the original event can be expressed as the union of these three (disjoint) events.

$$\begin{aligned}
P(N_8 - N_3 = 7 | N_5 - N_4 = 5) &= P(N_8 - N_5 = 0 \cap N_4 - N_3 = 2) \\
&\quad + P(N_8 - N_5 = 1 \cap N_4 - N_3 = 1) \\
&\quad + P(N_8 - N_5 = 2 \cap N_4 - N_3 = 0) \qquad (1) \\
&= P(N_8 - N_5 = 0)P(N_4 - N_3 = 2) \\
&\quad + P(N_8 - N_5 = 1)P(N_4 - N_3 = 1) \\
&\quad + P(N_8 - N_5 = 2)P(N_4 - N_3 = 0) \qquad (2) \\
&= (e^{-3\lambda}\frac{(3\lambda)^0}{0!})(e^{-\lambda}\frac{(\lambda)^2}{2!}) \\
&\quad + (e^{-3\lambda}\frac{(3\lambda)^1}{1!})(e^{-\lambda}\frac{(\lambda)^1}{1!}) \\
&\quad + (e^{-3\lambda}\frac{(3\lambda)^2}{2!})(e^{-\lambda}\frac{(\lambda)^0}{0!}) \qquad (3) \\
&= e^{-4\lambda}(\frac{\lambda^2}{2} + 3\lambda^2 + \frac{9\lambda^2}{2}) \qquad (4) \\
&= e^{-4\lambda}(8\lambda^2) \qquad (5)
\end{aligned}$$

You can note that this is equal to $e^{-4\lambda}(\frac{(4\lambda)^2}{2!})$ which is the probability of a count of 2 occurring in a time interval of 4.

(d) Find $P(N_5 - N_4 = 5 | N_8 - N_3 = 7)$.

Solution: We can use Bayes' Theorem based on our previous answers.

$$
\begin{aligned}
P(N_5 - N_4 = 5 | N_8 - N_3 = 7) &= \frac{P(N_8 - N_3 = 7 | N_5 - N_4 = 5) P(N_5 - N_4 = 5)}{P(N_8 - N_3 = 7)} && (6)\\
&= \frac{e^{-4\lambda}(\frac{(4\lambda)^2}{2!}) e^{-\lambda}(\frac{(\lambda)^5}{5!})}{e^{-5\lambda}(\frac{(5\lambda)^7}{7!})} && (7)\\
&= \frac{7!}{2!5!} \frac{(4\lambda)^2 \lambda^5}{(5\lambda)^7} && (8)\\
&= \binom{7}{2} \left(\frac{4}{5}\right)^2 \left(\frac{1}{5}\right)^5 && (9)\\
&\approx 0.0043 && (10)
\end{aligned}
$$

3. **[Communication in Gaussian Noise]**
A wireless communication system consists of a transmitter and a receiver. The transmitter sends a signal x, and the receiver observes

$$Y = x + Z,$$

where Z is a noise term, modeled as a Gaussian random variable with mean $\mu_Z = 0$ and variance $\sigma_Z^2 = 2$.

(a) Suppose the transmitted signal is $x = 1$. What is the pdf of the received signal Y?
Solution:

$$E[Y] = E[1 + Z] = 1 + E[Z] = 1 \tag{11}$$

$$
\begin{aligned}
\mathrm{Var}(Y) &= E[(Y - E[Y])^2] && (12)\\
&= E[(Z + 1 - 1)^2] = E[Z^2] && (13)\\
&= \mathrm{Var}(Z) + (E[Z])^2 = 2 && (14)
\end{aligned}
$$

The received signal Y is a linearly scaled version of a Gaussian RV Z. Therefore, it follows a Gaussian distribution with $\mu_Y = 1$ and $\sigma_Y^2 = 2$, i.e., a $\mathcal{N}(1, 2)$. The pdf is

$$f_Y(y) = \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{(y-1)^2}{4}\right)$$

(b) Now suppose the transmitted signal can be either $x = -1$ or $x = 1$. The receiver uses the following decoding rule: if $Y > 0$, it declares that $x = 1$; if $Y \le 0$, it declares that $x = -1$. Assuming that the transmitter sends -1 or $+1$ with probability $1/2$ each, what is the receiver's error probability in terms of Q? (P.S. This modulation is called BPSK)
Solution: The received signal $Y = Y_1 \sim \mathcal{N}(1, 2)$ when $x = 1$ and $Y = Y_{-1} \sim \mathcal{N}(-1, 2)$ when $x = -1$. Hence, the error probabilities for the two cases are given by: $P(Y \le 0 | x = 1)$ and $P(Y > 0 | x = -1)$ are given by:

$$
\begin{aligned}
P(Y \le 0 | x = 1) &= P(Y_1 < 0)\\
&= P(\frac{Y_1 - 1}{\sqrt{2}} < \frac{-1}{\sqrt{2}})\\
&= \Phi(\frac{-1}{\sqrt{2}}) = Q(\frac{1}{\sqrt{2}})
\end{aligned}
$$

$$\begin{aligned} P(Y > 0|x = -1) &= P(Y_{-1} > 0) \\ &= P(\frac{Y_{-1} + 1}{\sqrt{2}} > \frac{1}{\sqrt{2}}) = Q(\frac{1}{\sqrt{2}}) \end{aligned}$$

Hence, from the law of total probability, the receiver's probability of error is

$$\begin{aligned} p_{error} &= P(x = 1)P(Y \leq 0|x = 1) + P(x = -1)P(Y > 0|x = -1) \\ &= Q(\frac{1}{\sqrt{2}})(\frac{1}{2} + \frac{1}{2}) = Q(\frac{1}{\sqrt{2}}) \end{aligned}$$

(c) Now suppose the transmitted signal x can be chosen from three possible values: $x = -2$, $x = 0$ and $x = 2$. The receiver now uses the following decoding rule: if $Y < -1$, it declares $x = -2$, if $Y > 1$, it declares $x = 2$, and otherwise it declares $x = 0$. Assuming the transmitter sends each possible symbol with probability 1/3, what is the receiver's error probability?

Solution: Similar to the previous part, we define Y_{-2}, Y_0 and Y_2 as

$$Y_2 \sim \mathcal{N}(2, 2)$$

$$Y_0 \sim \mathcal{N}(0, 2)$$

$$Y_{-2} \sim \mathcal{N}(-2, 2)$$

The individual probabilities of error are

$$\begin{aligned} P(Y \leq 1|x = 2) &= P(Y_2 \leq 1) \\ &= P(\frac{Y_2 - 2}{\sqrt{2}} \leq \frac{-1}{\sqrt{2}}) \\ &= \Phi(\frac{-1}{\sqrt{2}}) = Q(\frac{1}{\sqrt{2}}) \end{aligned}$$

$$\begin{aligned} P(Y > 1 \cup Y < -1|x = 0) &= P(Y_0 > 1 \cup Y_0 < -1) = P(Y_0 > 1) + P(Y_0 < -1) \\ &= P(\frac{Y_0}{\sqrt{2}} > \frac{1}{\sqrt{2}}) + P(\frac{Y_0}{\sqrt{2}} < -\frac{1}{\sqrt{2}}) \\ &= Q(\frac{1}{\sqrt{2}}) + \Phi(-\frac{1}{\sqrt{2}}) = 2Q(\frac{1}{\sqrt{2}}) \end{aligned}$$

$$\begin{aligned} P(Y \geq -1|x = -2) &= P(Y_{-2} \geq -1) \\ &= P(\frac{Y_{-2} + 2}{\sqrt{2}} \geq \frac{1}{\sqrt{2}}) \\ &= Q(\frac{1}{\sqrt{2}}) \end{aligned}$$

The receiver's probability of error is

$$
\begin{aligned}
p_{error} = &P(x=2)P(Y \le 1|x=2) + \\
&P(x=0)P(Y > 1 \cup Y < -1|x=0) + \\
&P(x=-2)P(Y \ge -1|x=-2) \\
&= Q(\frac{1}{\sqrt{2}})(\frac{1}{3} + \frac{2}{3} + \frac{1}{3}) \\
&= \frac{4}{3}Q(\frac{1}{\sqrt{2}})
\end{aligned}
$$

4. **[Scaling of distributions]**

   (a) Assume that you have a random number generator that produces a uniformly distributed random variable X over the interval $[-2, 6]$. Find a linear function g such that $g(X) = Y$, where Y is uniformly distributed over the interval $[2, 10]$.

   Solution: Since X and Y have the same type of distribution, we can express

$$Y = aX + b.$$

   Given $X \sim \text{Unif}[-2, 6]$, we know

$$\mathbb{E}[X] = 2, \qquad \text{Var}(X) = \frac{(6-(-2))^2}{12} = \frac{64}{12} = \frac{16}{3}.$$

   Similarly, given $Y \sim \text{Unif}[2, 10]$, we have

$$\mathbb{E}[Y] = 6, \qquad \text{Var}(Y) = \frac{(10-2)^2}{12} = \frac{64}{12} = \frac{16}{3}.$$

   Furthermore, since $Y = aX + b$, it follows that

$$\mathbb{E}[Y] = a\mathbb{E}[X] + b = 6,$$

   and

$$\text{Var}(Y) = a^2 \text{Var}(X) = \frac{16}{3}.$$

   From the variance equation, we get $a^2 = 1 \Rightarrow a = \pm 1$. Substituting into the expectation equation gives

$$6 = a(2) + b.$$

   Hence:

$$\begin{cases} a = 1 \Rightarrow b = 4, \\ a = -1 \Rightarrow b = 8. \end{cases}$$

   Therefore, one can generate samples of $Y \sim \text{Unif}[2, 10]$ from $X \sim \text{Unif}[-2, 6]$ by using either

$$Y = X + 4, \quad \text{or} \quad Y = -X + 8.$$

(b) Solve part (a) with $X \sim \mathcal{N}(-2, 4)$ and $Y \sim \mathcal{N}(1, 2)$.

Solution: Since $Y = aX + b$ is a linear transformation of X, the resulting random variable Y is also normally distributed:

$$Y \sim \mathcal{N}(a\mu_X + b,\, a^2\sigma_X^2),$$

where μ_X and σ_X^2 are the mean and variance of X.

Given:

$$\mu_X = -2, \quad \sigma_X^2 = 4, \quad \mu_Y = 1, \quad \sigma_Y^2 = 2.$$

We match the mean and variance:

$$\begin{cases} a\mu_X + b = \mu_Y, \\ a^2\sigma_X^2 = \sigma_Y^2. \end{cases}$$

Substituting the given parameters:

$$\begin{cases} a(-2) + b = 1, \\ a^2(4) = 2. \end{cases}$$

From the second equation,

$$a^2 = \frac{1}{2} \quad \Rightarrow \quad a = \pm\frac{1}{\sqrt{2}}.$$

Substitute into the first equation to solve for b:

$$-2a + b = 1 \quad \Rightarrow \quad b = 1 + 2a.$$

Thus, we obtain two possible linear transformations:

$$\boxed{\begin{array}{lll} & a = \frac{1}{\sqrt{2}}, & b = 1 + \sqrt{2}, \\ \text{or} & a = -\frac{1}{\sqrt{2}}, & b = 1 - \sqrt{2}. \end{array}}$$

Therefore, samples from $Y \sim \mathcal{N}(1, 2)$ can be generated from samples of $X \sim \mathcal{N}(-2, 4)$ by using either transformation:

$$Y = \frac{X}{\sqrt{2}} + (1 + \sqrt{2}) \quad \text{or} \quad Y = -\frac{X}{\sqrt{2}} + (1 - \sqrt{2}).$$