ECE 313: Problem Set 7: Problems and Solutions

1. Let U_n denote the time it took to finish the nth round. We have $U_n \sim \exp(\lambda)$. Let $T = U_1 + U_2 + \cdots + U_{10}$. T has Erlang distribution with parameter $(10, \lambda)$. Our observation is $T = 5$. We want to pick $\lambda > 0$ so that $f_T(5)$ is maximized:

$$\hat{\lambda}_{ML} = \arg\max_{\lambda > 0} f_T(5) = \arg\max_{\lambda > 0} \frac{\lambda e^{-5\lambda} (5\lambda)^9}{9!}$$

We need to solve

$$\frac{df_T(5)}{d\lambda} = 0 \implies (-5) e^{-5\lambda} \lambda^{10} + 10\lambda^9 e^{-5\lambda} = 0 \implies -5\lambda + 10 = 0 \implies \hat{\lambda}_{ML} = 2$$

2. Let X denote the number of refunds that are issued.

(a) $P(X \geq 6) = \sum_{k \geq 6} \binom{n}{k} p^k (1-p)^{n-k} = 1 - \sum_{k \leq 5} \binom{10^6}{k} \left(2 \cdot 10^{-6}\right)^k \left(1 - 2 \cdot 10^{-6}\right)^{10^6 - k}$

(b) By Markov inequality

$$P(X \geq 6) \leq \frac{E[X]}{6} = \frac{np}{6} = \frac{1}{3}$$

(c) By Chebyshev's inequality

$$P(|X - np| \geq a\sigma) \leq \frac{1}{a^2}$$

Notice that $np = 2 \cdot 10^{-6} \cdot 10^6 = 2$ and $\sigma = \sqrt{np(1-p)} \approx \sqrt{2}$. Hence if we pick $a = 4/\sqrt{2}$, we get

$$P(|X - 2| \geq 4) \leq \left(\frac{\sqrt{2}}{4}\right)^2$$

Notice that if $|X - 2| \geq 4$ then $X \geq 6$ or $X \leq -2$; however X is non-negative. Hence,

$$P(X \geq 6) = P(|X - 2| \geq 4) \leq \frac{1}{8},$$

which is much sharper.

(d) Let $\lambda = np = 2$. We have

$$P(X \geq 6) = 1 - P(X \leq 5) \approx 1 - \sum_{k=0}^{5} \frac{e^{-2} 2^k}{k!}$$

(e) Notice that $\mu = np = 2$ and $\sigma = \sqrt{np(1-p)} \approx \sqrt{2}$. Let $\widetilde{X}$ denote the normal approximation.

$$P(X \geq 6) \approx P\left(\widetilde{X} \geq 5.5\right) = P\left(\frac{\widetilde{X} - 2}{\sqrt{2}} \geq \frac{5.5 - 2}{\sqrt{2}}\right) = Q\left(\frac{5.5 - 2}{\sqrt{2}}\right) = 1 - \Phi\left(\frac{3.5}{\sqrt{2}}\right)$$

(f) We observe $X = 20$. Notice that $P(X = 20) \approx P\left(\widetilde{X} \leq 20.5\right) - P\left(\widetilde{X} \leq 19.5\right)$ where $X \sim \mathcal{N}(\mu, \sigma^2), \mu = np$ and $\sigma = \sqrt{np(1-p)} \approx \sqrt{np} = \sqrt{\mu}$. Hence $\widetilde{X} \sim \mathcal{N}(\mu, \mu)$. We can pick μ that maximizes right hand side and then set $\hat{P}_{ML} = \hat{\mu}_{ML} \cdot 10^{-6}$.

$$\arg\max_{\mu > 0} P\left(\widetilde{X} \leq 20.5\right) - P\left(\widetilde{X} \leq 19.5\right) = \arg\max_{\mu > 0} \Phi\left(\frac{20.5 - \mu}{\sqrt{\mu}}\right) - \Phi\left(\frac{19.5 - \mu}{\sqrt{\mu}}\right)$$

Taking derivative and setting it to be zero, we get

$$\left(-\frac{20.5\mu^{-\frac{3}{2}}}{2}-\frac{\mu^{-\frac{1}{2}}}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(20.5-\mu)^2}{2\mu}\right)-\left(-\frac{19.5\mu^{-\frac{3}{2}}}{2}-\frac{\mu^{-\frac{1}{2}}}{2}\right)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(19.5-\mu)^2}{2\mu}\right)=0$$

$$\Longrightarrow \left(\frac{20.5}{\mu}+1\right)\exp\left(-\frac{(20.5-\mu)^2}{2\mu}\right)-\left(\frac{19.5}{\mu}+1\right)\exp\left(-\frac{(19.5-\mu)^2}{2\mu}\right)=0$$

$$\Longrightarrow \frac{\mu+20.5}{\mu+29.5}=\exp\left[\frac{1}{2\mu}\left((20.5-\mu)^2-(19.5-\mu)^2\right)\right]=\exp\left[\frac{1}{2\mu}(40-2\mu)\right]$$

$$\Longrightarrow \hat{\mu}_{ML}\approx 19.5062$$

$$\Longrightarrow \hat{p}_{ML}=10^{-6}\cdot 19.5062$$

3.

(a) $P(X \geq 50)=\Phi\left(-\frac{50-\mu}{\sigma}\right)=\Phi\left(-\frac{50-39.1}{5}\right)$

(b) $P(30 \leq X \leq 50)=P(X\geq 30)-P(X\geq 50)=\Phi\left(-\frac{30-39.1}{5}\right)-\Phi\left(-\frac{50-39.1}{5}\right)$

(c) We need to pick k so that $P(X\geq k)=0.9$, which means $\Phi\left(-\frac{k-39.1}{5}\right)\approx 0.9$.