ECE 313: Problem Set 5: Problems and Solutions

Due: *Tuesday* September 27 at 11:59 p.m.
Reading: *ECE 313 Course Notes*, Sections 2.11

1. **[House phone number]**
You move into a new house; the phone is connected. You are sure that the first 6 digits are 217-383, but not certain about the last 4 digits. You think the phone number might be 217-383-3428. As an experiment, you pick up the phone and dial 217-383-3428 at 6am. You obtain a busy signal. Note that the phone line will be busy if you call the same phone. For invalid numbers, you get an error message instead of busy line signal. Suppose that the total number of valid phone numbers that start from 217-383 in Champaign IL is 500. Are you now more sure of your phone number? If so, how much? What assumptions do you make to reach the conclusion?

Solution: There are two hypothesis:

- H_0: your phone number is 217-383-3428;
- H_1: it is another number

The data D is "when I call the number at 6am, I got a busy signal." The probability of D given H_0 is $P(D|H_0) = 1$. On the other hand, if H_1 is true, then the probability that the number dialed returns a busy signal is smaller than 1. The probability α that random 4 digit number is valid is about $\alpha = 500/10^4 = 0.05$. The probability β that you get a busy signal when you dial a valid phone number is equal to the fraction of phones you think are in use or off-the-hook when you make your tentative call. This fraction varies from town to town and with the time of day. Here we make an assumption that about 1% or fewer would be in use at 6am.

Note that the probability $P(D|H_1)$ is the product of α and β.

$$P(D|H_1) \approx 0.05 \cdot 0.01 = 5 \cdot 10^{-4}$$
$$\frac{P(H_1|D)}{P(H_0|D)} = \frac{P(D|H_1)\,P(H_1)}{P(D|H_0)\,P(H_0)}$$

Here we make another assumption on the prior probability $P(H_0) = P(H_1) = 0.5$. Then

$$P(H_0|D) = \frac{1}{1 + \frac{P(H_1|D)}{P(H_0|D)}} \approx 0.9995.$$

Note for the graders: the actual numbers can vary based on the student's assumptions.

2. **[Matching Bernoulli parameters]**
Consider hypotheses H_0 and H_1 about a two dimensional observation vector $X = (X_1, X_2)$. Under H_0, X_1 and X_2 are independent and identically distributed. Both have the Bernoulli distribution with $p = 0.5$. Under H_1, X_1 and X_2 are mutually independent, X_1 has the Bernoulli distribution with mean $p = 0.2$, and X_2 has the Bernoulli distribution with mean $p = 0.8$.

(a) Describe the maximum likelihood rule for deciding which hypothesis is true.

(b) Describe the MAP rule for deciding which hypothesis is true, assuming the prior distribution with $\frac{\pi_0}{\pi_1} = \frac{1}{2}$.

Solution: Possible outcomes for $X = (X_1, X_2)$ are: 00, 10, 01, 11.

$$
\begin{aligned}
P_1(00) &= (1-0.2)(1-0.8) = 0.16 \\
P_1(10) &= 0.2 \cdot 0.2 = 0.04 \\
P_1(01) &= (1-0.2) \cdot 0.8 = 0.64 \\
P_1(11) &= 0.2 \cdot 0.8 = 0.16 \\
P_0(00) &= P_0(11) = P_0(10) = P_0(01) = 0.25
\end{aligned}
$$

The likelihood ratio $\Gamma(k)$ is:

$$
\begin{aligned}
\frac{P_1(00)}{P_0(00)} &= \frac{16}{25} \\
\frac{P_1(10)}{P_0(10)} &= \frac{4}{25} \\
\frac{P_1(01)}{P_0(01)} &= \frac{64}{25} \\
\frac{P_1(11)}{P_0(11)} &= \frac{16}{25}.
\end{aligned}
$$

(a) ML decision rule: $H_1 : \Lambda(k) > 1$, $H_0 : \Lambda(k) < 1$
Declare H_1 if $X_1 = 0$ and $X_2 = 1$. Declare H_0 if $X = (X_1, X_2) \in \{(0,0), (1,0), (1,1)\}$

(b) MAP decision rule: $H_1 : \Lambda(k) > \tau$, $H_0 : \Lambda(k) < \tau$ where $\tau = \frac{\pi_0}{\pi_1} = \frac{1}{2}$.

- Declare H_1 if $X = (X_1, X_2) \in \{(0,0), (0,1), (1,1)\}$.
- Declare H_0 if $X = (X_1, X_2) \in \{(1,0)\}$.

3. **[A bent coin]**
Suppose you keep flipping a coin until you observe 3 heads. The random variable X is the number of flips that is required. Based on the observation, you need to choose one of the following two hypothesis: H_0: it is a fair coin with $P(H) = 0.5$, and H_1: the coin is bent with $P(H) = \frac{2}{3}$.

(a) Describe the ML decision rule. Express it in a simplified form. (Hint: $\frac{\log 8}{\log 1.5} = 5.13$.)

(b) Describe the MAP decision rule under the assumption that H_0 is a priori twice as likely as H_1. Express it in a simplified form. (Hint: $\frac{\log 16}{\log 1.5} = 6.84$.)

(c) Find the average error probability, p_e, for the ML rule, using the same prior distribution given in part (b).

(d) Find the average error probability, p_e, for the MAP rule, using the same prior distribution given in part (b).

Solution:

(a) Under H_1, $(r = 3)$

$$
P_1(n) = \binom{n-1}{r-1} p^r (1-p)^{n-r} = \frac{(n-1)(n-2)}{2} \left(\frac{2}{3}\right)^3 \frac{1}{3}^{n-3}.
$$

Under H_0,

$$
P_0(n) = \frac{(n-1)(n-2)}{2} \left(\frac{1}{2}\right)^n.
$$

Therefore, the likelihood ratio is

$$
\Lambda(n) = \frac{P_1(n)}{P_0(n)} = \frac{\frac{2}{3}^3 \frac{1}{3}^{n-3}}{\frac{1}{2}^n} = 8 \cdot \frac{2}{3}^n
$$

The ML decision rule is when $\Lambda(n) > 1$, we declare H_1 and if $\Lambda(n) < 1$, we declare H_0. Therefore,

- Declare H_1 if $n \leq 5$
- Declare H_0 if $n \geq 6$

(b) $\frac{\pi_2}{\pi_1} = 2$. According to the MAP decision rule, if $\Lambda(n) > 2$, we declare H_1. In other words,

- Declare H_1 if $n = 3$
- Declare H_0 if $n \geq 4$

(c) Since $\pi_0 = 2\pi_1$, $\pi_0 = \frac{2}{3}$, $\pi_1 = \frac{1}{3}$. Under ML decision rule, if $k \leq 5$, then we declare H_1.

$$
\begin{aligned}
P_{false} &= \sum_{n:\text{declare } H_1} P_0(n) = \sum_{n=3}^{5} \frac{(n-1)(n-2)}{2} \left(\frac{1}{2}\right)^n = \frac{1}{2} \\
P_{miss} &= \sum_{n:\text{declare } H_0} P_1(n) = 8 \cdot \sum_{n=6}^{\infty} \frac{(n-1)(n-2)}{2} \left(\frac{1}{3}\right)^n \\
&= 1 - 8 \cdot \sum_{n=3}^{5} \frac{(n-1)(n-2)}{2} \left(\frac{1}{3}\right)^n = 1 - \frac{64}{81} = \frac{17}{81} \\
P_{e,ML} &= \pi_0 \cdot P_{false} + \pi_1 \cdot P_{miss} = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{17}{81} = \frac{98}{243}
\end{aligned}
$$

(d) Under MAP decision rule, if $k = 3$, then we declare H_1.

$$
\begin{aligned}
P_{false} &= \sum_{n:\text{declare } H_1} P_0(n) = \frac{(3-1)(3-2)}{2} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8} \\
P_{miss} &= \sum_{n:\text{declare } H_0} P_1(n) = 8 \cdot \sum_{n=4}^{\infty} \frac{(n-1)(n-2)}{2} \left(\frac{1}{3}\right)^n \\
&= 1 - 8 \cdot \frac{(3-1)(3-2)}{2} \left(\frac{1}{3}\right)^3 = 1 - \frac{8}{27} = \frac{19}{27} \\
P_{e,MAP} &= \pi_0 \cdot P_{false} + \pi_1 \cdot P_{miss} = \frac{2}{3} \cdot \frac{1}{8} + \frac{1}{3} \cdot \frac{19}{27} = \frac{103}{324}
\end{aligned}
$$

4. **[True or false questions]**
Consider a binary hypothesis testing problem with H_0: X follows a geometric distribution with parameter $p = 0.5$, and H_1: X follows a geometric distribution with parameter $p = 0.2$. Please state whether the following statements are true or false and provide reasoning.

(a) If the priors $\pi_0 = \pi_1$, then the ML and the MAP estimators are the same.

(b) If the ML decision rule is employed, then $p_{\text{false alarm}} > p_{\text{miss}}$.

(c) MAP decision rule always provides lower $p_{\text{false alarm}}$ than ML decision rule.

Solution:

(a) True. When $\pi_0 = \pi_1$, $\tau = \frac{\pi_0}{\pi_1} = 1$, the thresholds for ML and MAP decision rules are identical.

(b) False.

$$
\begin{aligned}
P_1(k) &= 0.2 \cdot (0.8)^{k-1} \\
P_0(k) &= (0.5)^k \\
\Lambda(k) &= \frac{P_1(k)}{P_0(k)} = \frac{\frac{1}{4} \cdot \left(\frac{4}{5}\right)^k}{\left(\frac{1}{2}\right)^k} = \frac{1}{4} \cdot \left(\frac{8}{5}\right)^k
\end{aligned}
$$

When $k \geq 3$, $\Lambda(k) > 1$, declare H_1. For $k = 1, 2$ $\Lambda(k) < 1$, declare H_0.

$$P_{false} = \sum_{k=3}^{\infty} P_0(k) = \sum_{k=3}^{\infty} \left(\frac{1}{2}\right)^k = \frac{\left(\frac{1}{2}\right)^3}{1 - \frac{1}{2}} = \frac{1}{4} = 0.25$$

$$P_{miss} = \sum_{k=1}^{2} P_1(k) = 0.2 \cdot (1 + 0.8) = 0.36$$

Therefore, $P_{false} < P_{miss}$.

(c) False. MAP decision rule gives lower probability of average error depends on π_0, π_1, $P_e = \pi_0 \cdot P_{false} + \pi_1 \cdot P_{miss}$. MAP can result in larger P_{false}.