1. **[House phone number]**

You move into a new house; the phone is connected. You are sure that the first 6 digits are 217-383, but not certain about the last 4 digits. You think the phone number might be 217-383-3428. As an experiment, you pick up the phone and dial 217-383-3428 at 6am. You obtain a busy signal. Note that the phone line will be busy if you call the same phone. For invalid numbers, you get an error message instead of busy line signal. Suppose that the total number of valid phone numbers that start from 217-383 in Champaign IL is 500. Are you now more sure of your phone number? If so, how much? What assumptions do you make to reach the conclusion?

**Solution:**

There are two hypothesis:

- $H_0$: your phone number is 217-383-3428;
- $H_1$: it is another number

The data $D$ is "when I call the number at 6am, I got a busy signal." The probability of $D$ given $H_0$ is $P(D|H_0) = 1$. On the other hand, if $H_1$ is true, then the probability that the number dialed returns a busy signal is smaller than 1. The probability $\alpha$ that random 4 digit number is valid is about $\alpha = \frac{500}{10^4} = 0.05$. The probability $\beta$ that you get a busy signal when you dial a valid phone number is equal to the fraction of phones you think are in use or off-the-hook when you make your tentative call. This fraction varies from town to town and with the time of day. Here we make an assumption that about 1% or fewer would be in use at 6am.

Note that the probability $P(D|H_1)$ is the product of $\alpha$ and $\beta$.

$$P(D|H_1) \approx 0.05 \cdot 0.01 = 5 \cdot 10^{-4}$$

$$P(H_1|D) = \frac{P(D|H_1) P(H_1)}{P(H_0|D)} = \frac{P(D|H_1) P(H_1)}{P(D|H_0) P(H_0)}$$

Here we make another assumption on the prior probability $P(H_0) = P(H_1) = 0.5$. Then

$$P(H_0|D) = \frac{1}{1 + \frac{P(H_1|D)}{P(H_0|D)}} \approx 0.9995.$$ 

Note for the graders: the actual numbers can vary based on the student’s assumptions.

2. **[Matching Bernoulli parameters]**

Consider hypotheses $H_0$ and $H_1$ about a two dimensional observation vector $X = (X_1, X_2)$. Under $H_0$, $X_1$ and $X_2$ are independent and identically distributed. Both have the Bernoulli distribution with $p = 0.5$. Under $H_1$, $X_1$ and $X_2$ are mutually independent, $X_1$ has the Bernoulli distribution with mean $p = 0.2$, and $X_2$ has the Bernoulli distribution with mean $p = 0.8$.

(a) Describe the maximum likelihood rule for deciding which hypothesis is true.

(b) Describe the MAP rule for deciding which hypothesis is true, assuming the prior distribution with $\pi_0 = \frac{1}{2}$.

**Solution:** Possible outcomes for $X = (X_1, X_2)$ are: 00, 10, 01, 11.
\[ P_1(00) = (1 - 0.2)(1 - 0.8) = 0.16 \]
\[ P_1(10) = 0.2 \cdot 0.2 = 0.04 \]
\[ P_1(01) = (1 - 0.2) \cdot 0.8 = 0.64 \]
\[ P_1(11) = 0.2 \cdot 0.8 = 0.16 \]
\[ P_0(00) = P_0(11) = P_0(10) = P_0(01) = 0.25 \]

The likelihood ratio \( \Gamma(k) \) is:

\[
\begin{align*}
P_1(00) / P_0(00) &= 16 / 25 \\
P_1(10) / P_0(10) &= 4 / 25 \\
P_1(01) / P_0(01) &= 64 / 25 \\
P_1(11) / P_0(11) &= 16 / 25 .
\end{align*}
\]

(a) ML decision rule: \( H_1 : \Lambda(k) > 1 \), \( H_0 : \Lambda(k) < 1 \)

- Declare \( H_1 \) if \( X_1 = 0 \) and \( X_2 = 1 \). Declare \( H_0 \) if \( X = (X_1, X_2) \in \{(0, 0), (1, 0), (1, 1)\} \)

(b) MAP decision rule: \( H_1 : \Lambda(k) > \tau \), \( H_0 : \Lambda(k) < \tau \) where \( \tau = \frac{\pi_0}{\pi_1} = \frac{1}{2} \)

- Declare \( H_1 \) if \( X = (X_1, X_2) \in \{(0, 0), (0, 1), (1, 1)\} \).
- Declare \( H_0 \) if \( X = (X_1, X_2) \in \{(1, 0)\} \).

3. **[A bent coin]**

Suppose you keep flipping a coin until you observe 3 heads. The random variable \( X \) is the number of flips that is required. Based on the observation, you need to choose one of the following two hypothesis:

- \( H_0 \): it is a fair coin with \( P(H) = 0.5 \), and \( H_1 \): the coin is bent with \( P(H) = \frac{2}{3} \).

(a) Describe the ML decision rule. Express it in a simplified form. (Hint: \( \log \frac{8}{\log 1.5} = 5.13 \).)

(b) Describe the MAP decision rule under the assumption that \( H_0 \) is a priori twice as likely as \( H_1 \). Express it in a simplified form. (Hint: \( \log \frac{16}{\log \frac{3}{2}} = 6.84 \).)

(c) Find the average error probability, \( p_e \), for the ML rule, using the same prior distribution given in part (b).

(d) Find the average error probability, \( p_e \), for the MAP rule, using the same prior distribution given in part (b).

**Solution:**

(a) Under \( H_1 \), \( \binom{n-1}{r-1} p^r (1-p)^{n-r} = \frac{(n-1)(n-2)}{2} \frac{2}{3}^3 \frac{1}{3}^{n-3} \)

Under \( H_0 \),

\[ P_0(n) = \frac{(n-1)(n-2)}{2} \left( \frac{1}{2} \right)^n \]

Therefore, the likelihood ratio is

\[ \Lambda(n) = \frac{P_1(n)}{P_0(n)} = \frac{2^3 \frac{1}{3}^{n-3}}{\frac{1}{2}^n} = 8 \cdot \frac{2^n}{3} \]

The ML decision rule is when \( \Lambda(n) > 1 \), we declare \( H_1 \) and if \( \Lambda(n) < 1 \), we declare \( H_0 \). Therefore,
• Declare $H_1$ if $n \leq 5$
• Declare $H_0$ if $n \geq 6$

(b) $\frac{\pi_0}{\pi_1} = 2$. According to the MAP decision rule, if $\Lambda(n) > 2$, we declare $H_1$. In other words,
• Declare $H_1$ if $n = 3$
• Declare $H_0$ if $n \geq 4$

(c) Since $\pi_0 = 2\pi_1$, $\pi_0 = \frac{2}{3}$, $\pi_1 = \frac{1}{3}$. Under ML decision rule, if $k \leq 5$, then we declare $H_1$.

$$P_{false} = \sum_{n: declare \ H_1} P_0(n) = \sum_{n=3}^{5} \frac{(n-1)(n-2)}{2} \left(\frac{1}{2}\right)^n = \frac{1}{2}$$

$$P_{miss} = \sum_{n: declare \ H_0} P_1(n) = 8 \sum_{n=6}^{\infty} \frac{(n-1)(n-2)}{2} \left(\frac{1}{3}\right)^n$$

$$= 1 - 8 \sum_{n=3}^{5} \frac{(n-1)(n-2)}{2} \left(\frac{1}{3}\right)^n = 1 - \frac{64}{81} = \frac{17}{81}$$

$$P_{e,ML} = \pi_0 \cdot P_{false} + \pi_1 \cdot P_{miss} = \frac{2}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{17}{81} = \frac{98}{243}$$

(d) Under MAP decision rule, if $k = 3$, then we declare $H_1$.

$$P_{false} = \sum_{n: declare \ H_1} P_0(n) = \frac{(3-1)(3-2)}{2} \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$P_{miss} = \sum_{n: declare \ H_0} P_1(n) = 8 \sum_{n=4}^{\infty} \frac{(n-1)(n-2)}{2} \left(\frac{1}{3}\right)^n$$

$$= 1 - 8 \frac{(3-1)(3-2)}{2} \left(\frac{1}{3}\right)^3 = 1 - \frac{8}{27} = \frac{19}{27}$$

$$P_{e,MAP} = \pi_0 \cdot P_{false} + \pi_1 \cdot P_{miss} = \frac{2}{3} \cdot \frac{1}{8} + \frac{1}{3} \cdot \frac{19}{27} = \frac{324}{103}$$

4. [True or false questions]
Consider a binary hypothesis testing problem with $H_0$: $X$ follows a geometric distribution with parameter $p = 0.5$, and $H_1$: $X$ follows a geometric distribution with parameter $p = 0.2$. Please state whether the following statements are true or false and provide reasoning.

(a) If the priors $\pi_0 = \pi_1$, then the ML and the MAP estimators are the same.
(b) If the ML decision rule is employed, then $p_{false \ alarm} > p_{miss}$.
(c) MAP decision rule always provides lower $p_{false \ alarm}$ than ML decision rule.

Solution:

(a) True. When $\pi_0 = \pi_1$, $\tau = \frac{\pi_0}{\pi_1} = 1$, the thresholds for ML and MAP decision rules are identical.
(b) False.

$$P_1(k) = 0.2 \cdot (0.8)^{k-1}$$

$$P_0(k) = (0.5)^k$$

$$\Lambda(k) = \frac{P_1(k)}{P_0(k)} = \frac{\frac{1}{3} \cdot \left(\frac{4}{5}\right)^k}{\left(\frac{1}{2}\right)^k} = \frac{1}{4} \cdot \left(\frac{8}{5}\right)^k$$
When \( k \geq 3, \Lambda (k) > 1 \), declare \( H_1 \). For \( k = 1, 2 \Lambda (k) < 1 \), declare \( H_0 \).

\[
\begin{align*}
P_{\text{false}} &= \sum_{k=3}^{\infty} P_0 (k) = \sum_{k=3}^{\infty} \left( \frac{1}{2} \right)^k = \frac{\left( \frac{1}{2} \right)^3}{1 - \frac{1}{2}} = \frac{1}{4} = 0.25 \\
P_{\text{miss}} &= \sum_{k=1}^{2} P_1 (k) = 0.2 \cdot (1 + 0.8) = 0.36
\end{align*}
\]

Therefore, \( P_{\text{false}} < P_{\text{miss}} \).

(c) False. MAP decision rule gives lower probability of average error depends on \( \pi_0, \pi_1, P_e = \pi_0 \cdot P_{\text{false}} + \pi_1 \cdot P_{\text{miss}} \). MAP can result in larger \( P_{\text{false}} \).