University of Illinois **Fall 2022**

ECE 313: Problem Set 4

Due: *Monday* September 19 at 11:59 p.m.
Reading: *ECE 313 Course Notes*, Sections 2.8–2.10.

1. [**Circuit-design Department**]
   A production line decided to use the circuit design in Fig. 1. This design consists of three Integrated Circuits (**IC**s) $\{A, B, C\}$. Each IC *malfunctions with probability* $0 < p < 1$ independent of everything else, and a malfunctioning IC won't pass any current.

   We tested 100 kits from this manufacturer, and ten kits failed to pass any current, i.e., the current was unable to pass neither through "A and B" nor through "C".

   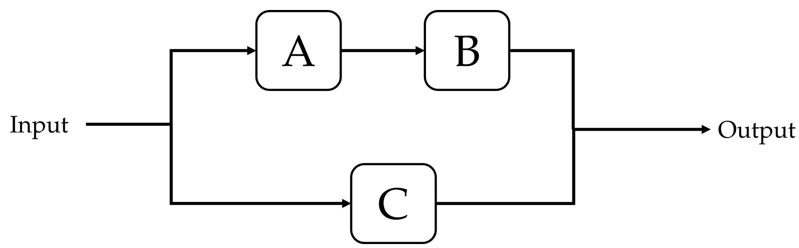

   Figure 1: Circuit design

   (a) [15 points] What is your best estimate of $\hat{p}_{\text{ML}}$? (*Hint*: roots of $1 - 20x^2 + 10x^3 = 0$ are approximately 1.97, -0.21, and 0.24.)

   (b) [10 points] We tested another kit, and it passed the current. Assuming your maximum-likelihood estimate of p is sharp, what is the probability that "C" is malfunctioning?

2. [**Airline industries**]
   Each airplane has capacity for 200 passengers, and overbooking is a common practice in these industries.

   (a) [10 points] To sell more tickets than available seats, the airline needs to estimate the probability that each passenger will attend the flight. Suppose that each passenger will attend the flight with probability p. The airline uses $\frac{X}{n}$ as the estimate of p, where n is the number of sold tickets and X is the number of people who attended the flight. How large n should be to estimate p within 0.1 with confidence of 0.99?

   (b) [20 points] According to the historical data, each passenger will attend a flight with probability $p_{\text{attend}} = 0.9$. What is the maximum number of tickets the airline can sell to ensure that no one is left behind with probability 0.75? (*Hint*: Use Chebychev's inequality, roots of $0.9x^2 + 0.6x - 200 = 0$ are -15.24 and 14.58)

3. [**Law of total probability and Bayes' rule**]
   Prove the law of total probability and Bayes' rule for conditional probability. Suppose that each event in the following expressions have a positive probability.

   (a) [10 points] Given a partition $E_1, E_2, \cdots, E_n$ of Ω, prove that $P(A|C) = \sum_{i=1}^{n} P(E_i|C)P(A|C \cap E_i)$.

   (b) [10 points] Prove that
   $$P(B|A, C) = \frac{P(A|B, C)P(B|C)}{P(A|C)}.$$

4. [**A forgetful mathematician**]
   A mathematician wakes up in front of his house, and he has no recollection of the events of last night.

He has lost his tablet, with all his papers on it! According to his schedule written on his hand, he went to the library first and then went out with his friends. So a reasonable hypothesis is to assume the tablet is either in the library or in his friends' car.

Suppose that the tablet is in the library with probability 0.8, and in the car with probability 0.2. If he searches the library, he will find the tablet with probability 0.1; if he searches his friend's car, he will find it with probability 0.9. Searching the library takes 10 minutes, and searching the car takes 5 minutes. He can search each place multiple times.

(a) [15 points] What is the best strategy for the mathematician to find his tablet in the first 10 minutes?

(b) [10 points] Suppose the mathematician calls his friend, and he confirms there is no tablet in his car. Suppose that his friend's search is successful with probability 0.7 if the tablet is in the car. What is the probability that the mathematician will find the tablet in his friend's car after searching it?