University of Illinois **Fall 2022**

ECE 313: Problem Set 3

Due: *Monday* September 12 at 11:59 p.m.
Reading: *ECE 313 Course Notes*, Sections 2.4.3–2.7

1. **[Conditional Probability and Independence]**

   (a) Consider events A, B, C related to a particular random experiment. **True or False**: If $P(A|C) > P(B|C)$ and $P(A|C^c) > P(B|C^c)$ then $P(A) > P(B)$.

   (b) Consider tossing a fair coin independently twice. Define the events:
   - $A = \{\text{the outcome of the first toss is } T\}$,
   - $B = \{\text{the outcome of the second toss is } T\}$,
   - $C = \{\text{the outcome of both tosses is the same}\}$.

   Are A, B, C pairwise independent? Are they independent?

2. **[Binomial Distribution]**
   Let X be a binomial random variable with parameters n and p. Use LOTUS to show that

$$E\left[\frac{1}{X+1}\right] = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

3. **[Poisson Distribution]**
   Let X be a Poisson random variable with parameter λ.

   (a) Show that

$$P(X = 2k, k = 0, 1, 2, \ldots) = \frac{1}{2}[1 + e^{-2\lambda}].$$

   (b) **True or False**: $E[X^3] = \lambda E[(X+1)^2]$.

4. **[Probability in Games]**
   Suppose that a roulette wheel consists of the numbers $\{0, 1, 2, \ldots, 36\}$ and the additional '00'. Suppose that you always bet that the outcome will be in the set $\{1, 2, \ldots, 15\}$. What is the probability that

   (a) you lose your first 3 bets.

   (b) your first win occurs on the 5th bet.

5. **[Theoretical Extensions]**
   Consider performing independent trials, each with probability of success $p \in (0, 1)$, until accumulating k successes. Let X be the corresponding number of trials until then. It can be shown that

$$P(X = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}, \quad n = k, k+1, \ldots$$

   In this setting, what is the probability of k successes occurring before m failures?