1. [Conditional Probability and Independence]

(a) Consider events $A, B, C$ related to a particular random experiment. **True or False:** If $P(A|C) > P(B|C)$ and $P(A|C^c) > P(B|C^c)$ then $P(A) > P(B)$.

(b) Consider tossing a fair coin independently twice. Define the events:
- $A = \{\text{the outcome of the first toss is } T\}$,
- $B = \{\text{the outcome of the second toss is } T\}$,
- $C = \{\text{the outcome of both tosses is the same}\}$.
Are $A, B, C$ pairwise independent? Are they independent?

2. [Binomial Distribution]
Let $X$ be a binomial random variable with parameters $n$ and $p$. Use LOTUS to show that
\[
E \left[ \frac{1}{X + 1} \right] = \frac{1 - (1 - p)^{n+1}}{(n+1)p}.
\]

3. [Poisson Distribution]
Let $X$ be a Poisson random variable with parameter $\lambda$.

(a) Show that
\[
P(X = 2k, k = 0, 1, 2, \ldots) = \frac{1}{2} [1 + e^{-2\lambda}] .
\]

(b) **True or False:** $E[X^3] = \lambda E[(X + 1)^2]$.

4. [Probability in Games]
Suppose that a roulette wheel consists of the numbers $\{0, 1, 2, \ldots, 36\}$ and the additional ‘00’. Suppose that you always bet that the outcome will be in the set $\{1, 2, \ldots, 15\}$. What is the probability that

(a) you lose your first 3 bets.

(b) your first win occurs on the 5th bet.

5. [Theoretical Extensions]
Consider performing independent trials, each with probability of success $p \in (0, 1)$, until accumulating $k$ successes. Let $X$ be the corresponding number of trials until then. It can be shown that
\[
P(X = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}, \quad n = k, k+1, \ldots
\]

In this setting, what is the probability of $k$ successes occurring before $m$ failures?