ECE 313: Problem Set 2

Due: *Tuesday* September 6 at 11:59 p.m.
Reading: *ECE 313 Course Notes*, Sections 2.1–2.4.2

1. **[Distribution of number of matches (35 points)]**
Suppose five people write their names on slips of paper; the slips of paper are randomly shuffled and then each person gets back one slip of paper; all possibilities of who gets what slip are equally likely. Let X denote the number of people who get back the slip with their own name.

   (a) (6 points) Describe the sample space Ω for the experiment. How many elements does it have?

   Suppose the people are assigned numbers 1 through 5. Then,

   $$\Omega = \{x_1x_2x_3x_4x_5 \mid x_1x_2x_3x_4x_5 \text{ is a permutation of } 12345\}$$

   where x_i represents the number given back to person i for $1 \le i \le 5$.

   For example, 15432 indicates person one gets slip 1, person two gets slip 5, person three gets slip 4, person four gets slip 3, and person five gets slip 2.

   $|\Omega| = 5! = 120$

   (b) (17 points) Find the probability mass function of X.

   The possible values of X are 0,1,2,3,5 (2 points)

   $\{X = 5\} = \{12345\}$ so $p_X(5) = 1/5! = 1/120$ (3 points)

   $\{X = 3\} = \{12354, 13245, \cdots\}$ Essentially, choose 3 numbers that match the remaining two are swapped. $p_X(3) = \binom{5}{3}/5! = 1/12$ (3 points)

   $\{X = 2\} = \{12534, 12453, 14352, 15324, \cdots\}$ Choose 2 numbers that match. Then the remaining 3 entries do not match. For the second part, the number of ways to order them equals 2.

   $p_X(2) = \binom{5}{2} \cdot 2/5! = 1/6$ (3 points)

   $\{X = 1\} = \{15432, 15423, 15234, 14523, 14532, 14253, 13254, 13524, 13452, \cdots\}$

   Choose one number that matches, and the remaining four entries do not match. The number of ways that four entries do not match is 9.

   Suppose $x_1 = 1$. Then for the four remaining people, there are three choices for x_2, and for each of those, there are three choices for which person to get slip 2.

   $p_X(1) = \binom{5}{1} \cdot 9/120 = 3/8$ (3 points)

   $p_X(0) = 1 - p_X(5) - p_X(4) - p_X(3) - p_X(2) - p_X(1) = 1 - 1/120 - 1/12 - 1/6 - 3/8 = 11/30$ (3 points)

   (c) (6 points) Find $E[X]$ and $E[X^2]$.

   $$E[X] = 1 \cdot \frac{3}{8} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{12} + 5 \cdot \frac{1}{120} = 1$$
   $$E[X^2] = 1 \cdot \frac{3}{8} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{12} + 25 \cdot \frac{1}{120} = 2.$$

   (d) (4 points) Find $\mathrm{Var}(X)$.

   $$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 2 - 1 = 1$$

   (e) (2 points) Find $\mathrm{Var}(2X + 1)$.

   $$\mathrm{Var}(2X + 1) = 4 \cdot \mathrm{Var}(X) = 4$$

2. **[Mean, Variance, LOTUS (15 points)]**
Consider a random variable X

(a) (4 points) Is it possible for the mean of X to be 5 and the standard deviation 0? If so, construct a pmf for X that corresponds to these values. If not, prove the impossibility.

$X = 5$ with probability 1 works. The pmf is concentrated at the value 5.

(b) (4 points) Is it possible for the mean of X to be 0, the second moment 1, and the standard deviation 1? If so, construct a pmf for X that corresponds to these values. If not, prove the impossibility.

Consider $X = \pm 1$ with probabilities 0.5 each.

(c) (7 points) Suppose the mean of X is μ and the standard deviation is σ. Find a formula for the mean of the random variable $Y = 3X^2 + 6X - 1$ in terms of μ and σ.

Note that $\sigma^2 = E[X^2] - \mu^2$.

$$\begin{aligned} E[Y] &= E[3X^2 + 6X - 1] \\ &= 3E[X^2] + 6E[X] - 1 \\ &= 3(\sigma^2 + \mu^2) + 6\mu - 1 \\ &= 3\sigma^2 + 3\mu^2 + 6\mu - 1 \end{aligned}$$

3. **[First and second moments of a ternary random variable (34 points)]**
This problem focuses on the possible mean and variance of a random variable X with support set $\{-1, 0, 1\}$. Let $p_X(-1) = a$ and $p_X(1) = b$ and $p_X(0) = 1 - a - b$. Let $\mu = E[X]$, $\sigma^2 = \mathrm{Var}(X)$, and $m_2 = E[X^2]$.

(a) (4 points) What are the conditions for a and b to make the pmf valid?

pmf is non-negative, meaning

$$\begin{aligned} &p_X(-1) = a \geq 0, \quad p_X(0) = b \geq 0, \quad p_X(1) = 1 - a - b \geq 0 \\ &\implies a \geq 0 \quad b \geq 0 \quad a + b \leq 1. \end{aligned}$$

(b) (10 points) Find (a, b) so that $\mu = \frac{1}{2}$ and $\sigma = \frac{1}{2}$.

Since $m_2 = \sigma^2 + \mu^2$, the given requirement is equivalent to $\mu = 1/2$ and $m_2 = 1/2$. In general, μ and m_2 can be expressed in terms of a and b as

$$\mu = b - a, \quad m_2 = a + b$$

Applying the given constraints we have

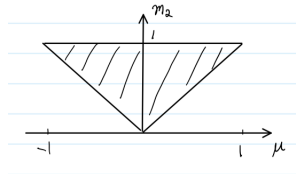
$$\mu = b - a = 1/2, \quad m_2 = a + b = 1/2 \quad \implies \quad a = 0, \quad b = 1/2$$

Note that this choice for (a, b) is valid since the conditions $a \geq 0, b \geq 0$, and $a + b \leq 1$ hold.

(c) (10 points) Express a and b in terms of μ and m_2. Determine and sketch the region of (μ, m_2) pairs for which there is a valid choice of (a, b).

Since $\mu = b - a$ and $m_2 = a + b$, $a = \frac{m_2 - \mu}{2}$ and $b = \frac{m_2 + \mu}{2}$.

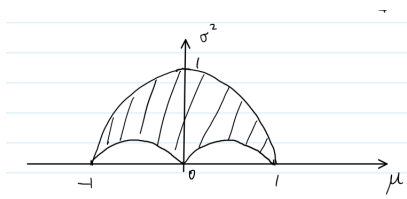
$$\begin{aligned} &a \geq 0 \implies \mu \leq m_2 \\ &b \geq 0 \implies \mu \geq -m_2 \implies |\mu| \leq m_2 \\ &a + b \leq 1 \implies m_2 \leq 1 \end{aligned}$$



(d) (10 points) Determine and sketch the set of (μ, σ^2) pairs for which there is a valid choice of (a, b).

From the support set, we know that the mean μ must be in the range $1 \leq \mu \leq -1$. Since $m_2 \leq 1$, we have $\sigma^2 = m_2 - \mu^2 \leq 1 - \mu^2$. Then

$$m_2 \geq |\mu| \implies \sigma^2 \geq |\mu| - \mu^2 = \begin{cases} \frac{1}{4} - (\mu - \frac{1}{2})^2, & \mu \geq 0 \\ \frac{1}{4} - (\mu + \frac{1}{2})^2, & \mu < 0 \end{cases}$$



4. **[Conditional probability (16 points)]**
Two fair dice are rolled.

(a) (8 points) Find the conditional probability that the difference is 2 given the product is odd.

Event A: the difference is 2.
Event B: the product is odd.

Event AB: $\{13, 35, 31, 53\}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{|AB|}{|B|} = \frac{4}{9}$$

(b) (8 points) Find the conditional probability that at least one die lands on 2, given that the product is even.

Event A: at least one die lands on 2.
Event B: the product is even $\Leftrightarrow$ one of the die is even.

Event AB: $\{12, 22, 32, 42, 52, 62, 21, 23, 24, 25, 26\}$

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{|AB|}{|B|} = \frac{11}{36 - 9} = \frac{11}{27}$$