ECE 313: Problem Set 2

Due: Tuesday September 6 at 11:59 p.m.
Reading: ECE 313 Course Notes, Sections 2.1–2.4.2

1. [Distribution of number of matches (35 points)]

Suppose five people write their names on slips of paper; the slips of paper are randomly shuffled and then each person gets back one slip of paper; all possibilities of who gets what slip are equally likely. Let $X$ denote the number of people who get back the slip with their own name.

(a) (6 points) Describe the sample space $\Omega$ for the experiment. How many elements does it have?

Suppose the people are assigned numbers 1 through 5. Then, $\Omega = \{x_1 x_2 x_3 x_4 x_5 \mid x_1 x_2 x_3 x_4 x_5 \text{ is a permutation of 12345}\}$

where $x_i$ represents the number given back to person $i$ for $1 \leq i \leq 5$.

For example, 15432 indicates person one gets slip 1, person two gets slip 5, person three gets slip 4, person four gets slip 3, and person five gets slip 2.

$|\Omega| = 5! = 120$

(b) (17 points) Find the probability mass function of $X$.

The possible values of $X$ are 0,1,2,3,5

\{X = 5\} = \{12345\} so $p_X(5) = 1/5! = 1/120$

\{X = 3\} = \{12345, 13245, \cdots\}$ Essentially, choose 3 numbers that match the remaining two are swapped. $p_X(3) = \binom{5}{3}/5! = 1/12$

\{X = 2\} = \{12534, 12453, 14352, 15324, \cdots\}$ Choose 2 numbers that match. Then the remaining 3 entries do not match. For the second part, the number of ways to order them equals 2.

$p_X(2) = \binom{5}{2} \cdot 2/5! = 1/6$

\{X = 1\} = \{15432, 15423, 15234, 14523, 14532, 14253, 13254, 13245, 13452, \cdots\}$

Choose one number that matches, and the remaining four entries do not match. The number of ways that four entries do not match is 9.

Suppose $x_1 = 1$. Then for the four remaining people, there are three choices for $x_2$, and for each of those, there are three choices for which person to get slip 2.

$p_X(1) = \binom{5}{1} \cdot 9/120 = 3/8$

$p_X(0) = 1 - p_X(5) - p_X(4) - p_X(3) - p_X(2) - p_X(1) = 1 - 1/120 - 1/12 - 1/6 - 3/8 = 11/30$

(c) (6 points) Find $E[X]$ and $E[X^2]$.

$E[X] = 1 \cdot \frac{3}{8} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{12} + 5 \cdot \frac{1}{120} = 1$

$E[X^2] = 1 \cdot \frac{3}{8} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{12} + 25 \cdot \frac{1}{120} = 2$

(d) (4 points) Find $\text{Var}(X)$.

$\text{Var}(X) = E[X^2] - (E[X])^2 = 2 - 1 = 1$

(e) (2 points) Find $\text{Var}(2X + 1)$.

$\text{Var}(2X + 1) = 4 \cdot \text{Var}(X) = 4$
2. [Mean, Variance, LOTUS (15 points)]
Consider a random variable $X$

(a) (4 points) Is it possible for the mean of $X$ to be 5 and the standard deviation 0? If so, construct a pmf for $X$ that corresponds to these values. If not, prove the impossibility.

$X = 5$ with probability 1 works. The pmf is concentrated at the value 5.

(b) (4 points) Is it possible for the mean of $X$ to be 0, the second moment 1, and the standard deviation 1? If so, construct a pmf for $X$ that corresponds to these values. If not, prove the impossibility.

Consider $X = \pm 1$ with probabilities 0.5 each.

(c) (7 points) Suppose the mean of $X$ is $\mu$ and the standard deviation is $\sigma$. Find a formula for the mean of the random variable $Y = 3X^2 + 6X - 1$ in terms of $\mu$ and $\sigma$.

Note that $\sigma^2 = E[X^2] - \mu^2$.

$$E[Y] = E[3X^2 + 6X - 1] = 3E[X^2] + 6E[X] - 1 = 3(\sigma^2 + \mu^2) + 6\mu - 1 = 3\sigma^2 + 3\mu^2 + 6\mu - 1$$

3. [First and second moments of a ternary random variable (34 points)]
This problem focuses on the possible mean and variance of a random variable $X$ with support set \{-1, 0, 1\}. Let $p_X(-1) = a$ and $p_X(1) = b$ and $p_X(0) = 1 - a - b$. Let $\mu = E[X]$, $\sigma^2 = \text{Var}(X)$, and $m_2 = E[X^2]$.

(a) (4 points) What are the conditions for $a$ and $b$ to make the pmf valid?

pmf is non-negative, meaning

$$p_X(-1) = a \geq 0, \quad p_X(0) = b \geq 0, \quad p_X(1) = 1 - a - b \geq 0$$

$\implies a \geq 0 \quad b \geq 0 \quad a + b \leq 1$.

(b) (10 points) Find $(a, b)$ so that $\mu = \frac{1}{2}$ and $\sigma = \frac{1}{2}$.

Since $m_2 = \sigma^2 + \mu^2$, the given requirement is equivalent to $\mu = 1/2$ and $m_2 = 1/2$. In general, $\mu$ and $m_2$ can be expressed in terms of $a$ and $b$ as

$$\mu = b - a, \quad m_2 = a + b$$

Applying the given constraints we have

$$\mu = b - a = 1/2, \quad m_2 = a + b = 1/2 \quad \implies a = 0, \quad b = 1/2$$

Note that this choice for $(a, b)$ is valid since the conditions $a \geq 0, b \geq 0,$ and $a + b \leq 1$ hold.

(c) (10 points) Express $a$ and $b$ in terms of $\mu$ and $m_2$. Determine and sketch the region of $(\mu, m_2)$ pairs for which there is a valid choice of $(a, b)$.

Since $\mu = b - a$ and $m_2 = a + b$, $a = \frac{m_2 - \mu}{2}$ and $b = \frac{m_2 + \mu}{2}$.

$$a \geq 0 \implies \mu \leq m_2$$

$$b \geq 0 \implies \mu \leq -m_2 \implies |\mu| \leq m_2$$

$$a + b \leq 1 \implies m_2 \leq 1$$
(d) (10 points) Determine and sketch the set of \((\mu, \sigma^2)\) pairs for which there is a valid choice of \((a, b)\).

From the support set, we know that the mean \(\mu\) must be in the range \(1 \leq \mu \leq -1\). Since \(m_2 \leq 1\), we have \(\sigma^2 = m_2 - \mu^2 \leq 1 - \mu^2\). Then

\[
m_2 \geq |\mu| \implies \sigma^2 \geq |\mu| - \mu^2 = \begin{cases} \frac{1}{4} - (\mu - \frac{1}{2})^2, & \mu \geq 0 \\ \frac{1}{4} - (\mu + \frac{1}{2})^2, & \mu < 0 \end{cases}
\]

4. [Conditional probability (16 points)]

Two fair dice are rolled.

(a) (8 points) Find the conditional probability that the difference is 2 given the product is odd.

Event A: the difference is 2.
Event B: the product is odd.
Event AB: \{13, 35, 31, 53\}

\[
P(A|B) = \frac{P(AB)}{P(B)} = \frac{|AB|}{|B|} = \frac{4}{9}
\]

(b) (8 points) Find the conditional probability that at least one die lands on 2, given that the product is even.

Event A: at least one die lands on 2.
Event B: the product is even ⇔ one of the die is even.
Event AB: \{12, 22, 32, 42, 52, 62, 21, 23, 24, 25, 26\}

\[
P(A|B) = \frac{P(AB)}{P(B)} = \frac{|AB|}{|B|} = \frac{11}{36 - 9} = \frac{11}{27}
\]